

Quantum deformations of $sl(2)$: The hard core of two dimensional gravity.

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Abstract

The quantum group structure of the Liouville theory is reviewed and shown to be a important tool for solving the theory.

1 Introduction

The Liouville theory arose more than ten years ago from Weyl anomaly in two dimensions. It is actually the simplest member of the family of conformal Toda theories[1] which should be considered as W gravity in the conformal gauge[2, 3]. Upon quantization quantum group structures emerge[4, 5, 7, 6] such that the mathematical parameters \hbar of the mathematical deformations coincide with the Planck constants, with appropriate unit choices. Thus, the non-commutativity which is inherent to the group deformation – since the co-product is non-symmetric – is brought about by the very quantization of these systems. Such a quantum group structure was already there from the beginning, in the early works of Neveu and myself[8], but in disguise. This appearance of quantum group seems to be very natural geometrically, since one deals with a gravity theory, where the space-time metric is quantized, which seems tantamount to quantizing the two dimensional (2D) space-time itself. Thus an object like a quantum plane should appear, and quantum

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groups are natural transformations of such “quantum manifolds”. In these lecture notes, we shall not follow this last line, but rather review the basic features of the Liouville theory, from the viewpoint taken[9, 10, 11, 12, 13] recently by E. Cremmer, J.-F. Roussel, J. Schnittger and myself, where the operator algebra of the chiral components is completely determined in terms of quantum group symbols, within the framework[14] of Moore and Seiberg. This result is instrumental in deriving the detailed properties of 2D gravity coupled with 2D matter. In the weak coupling regime of gravity, this gives back[19] the results of matrix models in a way which may seem unnecessarily painful. However, this quantum group approach is the only one which extends to the strong coupling regime. This recently allowed[13] J.-F. Roussel and myself to actually solve a new type of strongly coupled topological model, which we will mention at the end.

2 Basic points about Liouville theory

2.1 The classical case

In order to set the stage, we recall the classical structure for the special case of the Liouville theory. We shall work with Euclidean coordinates σ, τ . As a preparation for the quantum case, the classical action is defined as

$$S = \frac{1}{8\pi} \int d\sigma d\tau \left(\left(\frac{\partial\Phi}{\partial\sigma} \right)^2 + \left(\frac{\partial\Phi}{\partial\tau} \right)^2 + e^{2\sqrt{\gamma}\Phi} \right). \quad (2.1)$$

We use world sheet variables σ and τ , which are local coordinates such that the metric tensor takes the form $h_{ab} = \delta_{a,b} e^{2\Phi\sqrt{\gamma}}$. The complex structure is assumed to be such that the curves with constant σ and τ are everywhere tangent to the local imaginary and real axis respectively. The action 2.1 corresponds to a conformal theory such that $\exp(2\sqrt{\gamma}\Phi)d\sigma d\tau$ is invariant. It is convenient to let

$$x_{\pm} = \sigma \mp i\tau, \quad \partial_{\pm} = \frac{1}{2}(\partial_{\sigma} \pm i\partial_{\tau}). \quad (2.2)$$

By minimizing the above action, one derives the Liouville equations

$$\frac{\partial^2\Phi}{\partial\sigma^2} + \frac{\partial^2\Phi}{\partial\tau^2} = \sqrt{\gamma} e^{2\sqrt{\gamma}\Phi}. \quad (2.3)$$

The chiral modes may be separated very simply using the fact that the field $\Phi(\sigma, \tau)$ satisfies the above equation if and only if²

$$e^{-\sqrt{\gamma}\Phi} = \frac{i\sqrt{\gamma}}{2} \sum_{j=1,2} f_j(x_+)g_j(x_-); \quad x_{\pm} = \sigma \mp i\tau \quad (2.4)$$

where f_j (resp. g_j), which are functions of a single variable, are solutions of the same Schrödinger equation (primes mean derivatives)

$$-f_j'' + T(x_+)f_j = 0, \quad (\text{resp. } -g_j'' + \bar{T}(x_-)g_j). \quad (2.5)$$

The solutions are normalized such that their Wronskians $f_1'f_2 - f_1f_2'$ and $g_1'g_2 - g_1g_2'$ are equal to one. The proof goes as follows.

1) First check that Eq.2.4 is indeed solution. Taking the Laplacian of the logarithm of the right-hand side gives

$$\frac{\partial^2 \Phi}{\partial \sigma^2} + \frac{\partial^2 \Phi}{\partial \tau^2} \equiv 4\partial_+ \partial_- \Phi = -4\sqrt{\gamma} / \left(\sum_{i=1,2} f_i g_i \right)^2$$

where $\partial_{\pm} = (\partial/\partial\sigma \pm i\partial/\partial\tau)/2$. The numerator has been simplified by means of the Wronskian condition. This is equivalent to Eq.2.3.

2) Conversely check that any solution of Eq.2.3 may be put under the form Eq.2.4. If Eq.2.3 holds one deduces

$$\partial_{\mp} T^{(\pm)} = 0; \quad \text{with } T^{(\pm)} := e^{\sqrt{\gamma}\Phi} \partial_{\pm}^2 e^{-\sqrt{\gamma}\Phi} \quad (2.6)$$

$T^{(\pm)}$ are thus functions of a single variable. Next the equation involving $T^{(+)}$ may be rewritten as

$$(-\partial_+^2 + T^{(+)})e^{-\Phi} = 0 \quad (2.7)$$

with solution

$$e^{-\sqrt{\gamma}\Phi} = \frac{i\sqrt{\gamma}}{2} \sum_{j=1,2} f_j(x_+)g_j(x_-); \quad \text{with } -f_j'' + T^{(+)}f_j = 0$$

where the g_j are arbitrary functions of x_- . Using the equation 2.6 that involves $T^{(-)}$, one finally derives the Schrödinger equation $-g_j'' + T^{(-)}g_j = 0$. Thus the theorem holds with $T = T^{(+)}$ and $\bar{T} = T^{(-)}$. One may deduce from

² The factor i means that these solutions should be considered in Minkowsky space-time

Eq.2.6 that the potentials of the two Schrödinger equations coincide with the two chiral components of the stress-energy tensor. Thus these equations are the classical equivalent of the Ward identities that ensure the decoupling of Virasoro null vectors. Next a Bäcklund transformation to free fields is easily obtained by writing

$$f_1(x) = e^{q(x)\sqrt{\gamma}}, \quad f_2(x) = f_1(x) \int^x dx_1 e^{-2q(x_1)\sqrt{\gamma}}, \quad (2.8)$$

It follows from the canonical Poisson brackets of Liouville theory that

$$\{p(\sigma_1 - i\tau), p(\sigma_2 - i\tau)\}_{\text{PB}} = 2\pi\delta'(\sigma_1 - \sigma_2). \quad (2.9)$$

where $p = q'$. Clearly the differential equation Eq.2.5 gives $T = p^2 + p'/\sqrt{\gamma}$, or equivalently $T = (q')^2 + q''/\sqrt{\gamma}$. The last expression coincides with U_1 Sugawara stress-energy tensor with a linear term. From the viewpoint of differential equations, these relations are simply the well-known Riccati equations associated with the Schrödinger equation Eq.2.5. An easy computation shows that T satisfies the Poisson bracket Virasoro algebra with $C_{\text{class}} = 3/\gamma$. We shall consider the typical situation of a cylinder with $0 \leq \sigma \leq \pi$, and $-\infty \leq \tau \leq \infty$. After appropriate coordinate change, this may describe one handle of a surface with arbitrary genus. Then T is periodic in σ with period 2π , and it is important to discuss the monodromy properties of the Schrödinger equation, whose physics is that of one-dimensional crystal.

Consider any pair $f_{1,2}$ of independent solutions of the equations $-f'' + Tf = 0$. If $T(x + 2\pi) = T(x)$, one has $f_j(x + 2\pi) = \sum_k m_{jk} f_k(x)$, where the monodromy matrix m_{jk} is independent of x . In the generic case, the matrix is diagonalizable, and there exist the so-called Bloch waves solutions $V_{1,2}$ such that $V_j(x + 2\pi) = \alpha_j V_j(x)$. Since the Wronskian of the two solution is a constant, the product of eigenvalues $\alpha_1 \alpha_2$ is equal to one. We may identify one of the two Bloch waves, say, V_1 with f_1 if we take p to be periodic. Indeed if we write

$$p(x) = \sum_n e^{-inx} p_n$$

we have

$$q(x) = q_0 + p_0 x + i \sum_{n \neq 0} e^{-inx} p_n / n$$

and $V_1(x + 2\pi) = \exp(2\pi p_0) V_1(x)$. The other Bloch wave V_2 should be such that $V_2(x + 2\pi) = \exp(-2\pi p_0) V_2(x)$. This may be easily achieved by taking a

linear combination of f_1 and f_2 in Eq.2.8. As a preparation for the quantum case, let us remark that, at this level, there is a complete symmetry between V_1 and V_2 which are simply the two eigenvectors of the monodromy matrix (we do not consider the so-called parabolic situation where this matrix would not be diagonalizable). This is in apparent contrast with the expression Eq.2.8 where f_1 played a special role. We may re-establish this symmetry by writing

$$V_i = e^{q^{[i]}(x)\sqrt{\gamma}}, \quad p^{[i]} = q^{[i]'}, \quad i = 1, 2. \quad (2.10)$$

Since the V 's are solutions of the same Schrödinger equation, the fields $p^{[i]}$ are related by the equation

$$T = (p^{[1]})^2 + \frac{p^{[1]'}}{\sqrt{\gamma}} = (p^{[2]})^2 + \frac{p^{[2]'}}{\sqrt{\gamma}} \quad (2.11)$$

Thus the $p^{[i]}$'s are simply the two independent solutions of the Riccati equations associated with our Schrödinger equation 2.5. Of course $p^{[1]}$ coincides with the previous p field, and thus obey the P.B. algebra Eq.2.9. One may verify[15] that this is also true for $p^{[2]}$. Thus we have

$$\begin{aligned} \{p^{[1]}(\sigma_1 - i\tau), p^{[1]}(\sigma_2 - i\tau)\}_{\text{PB}} = \\ \{p^{[2]}(\sigma_1 - i\tau), p^{[2]}(\sigma_2 - i\tau)\}_{\text{PB}} = 2\pi\delta'(\sigma_1 - \sigma_2), \end{aligned} \quad (2.12)$$

Clearly we may expand each field in Fourier series:

$$q^{[j]}(x) = q_0^{[j]} + p_0^{[j]}x + i \sum_{n \neq 0} e^{-inx} p_n^{[j]}/n. \quad (2.13)$$

Since the corresponding eigenvalues of the monodromy matrix have their product equal to one, it follows that the $p_0^{[j]}$'s are related by

$$p_0^{[1]} = -p_0^{[2]}. \quad (2.14)$$

Eqs.2.11, and 2.14 define[15] a canonical transformation between two equivalent free fields. This structure is also present at the quantum level.

Let return finally to the Liouville field Eq.2.4. A priori, any two pairs of linearly independent solutions of Eq.2.5 are suitable. At this point it is convenient to introduce other solutions than the Bloch waves V_j , and \bar{V}_j just

discussed. We hereafter call \bar{f}_i the general pair describing the plus components. For the minus components, it is convenient to change the definition so that Eq.2.4 becomes

$$e^{-\sqrt{\gamma}\Phi} = \frac{i\sqrt{\gamma}}{2} \left(f_1(x_+) \bar{f}_2(x_-) - f_2(x_+) \bar{f}_1(x_-) \right); \quad (2.15)$$

With this convention, one easily sees that the Liouville exponential is left unchanged if f_j and \bar{f}_j are replaced by $\sum_k M_{jk} f_k$ and $\sum_k M_{kj} \bar{f}_k$, respectively, where M_{jk} is an arbitrary constant matrix with determinant equal to one. Eq.2.15 is $sl(2, C)$ -invariant with f_j transforming as a representation of spin $1/2$. At the quantum level, the f_j 's and \bar{f}_j 's will become operators that do not commute, and the group $sl(2)$ will get deformed to become the quantum group $U_q(sl(2))$. Note that the above Bloch-wave solutions V_i will not transform simply under action of the quantum group. This is why we introduced other pairs of solutions f , and \bar{f} . This structure plays a crucial role at the quantum level, and we now elaborate upon the classical $sl(2)$ structure where the calculations are simple. At the classical level, it is trivial to take Eq.2.15 to any power. For positive integer powers $2J$, one gets (letting $\beta = i\frac{\sqrt{\gamma}}{2}$)

$$e^{-2J\sqrt{\gamma}\Phi} = \sum_{M=-J}^J \frac{\beta^{2J}(-1)^{J-M}(2J)!}{(J+M)!(J-M)!} \left(f_1(x_+) \bar{f}_2(x_-) \right)^{J-M} \left(f_2(x_+) \bar{f}_1(x_-) \right)^{J+M}. \quad (2.16)$$

It is convenient to put the result under the form

$$e^{-2J\sqrt{\gamma}\Phi} = \beta^{2J} \sum_{M=-J}^J (-1)^{J-M} f_M^{(J)}(x_+) \bar{f}_{-M}^{(J)}(x_-). \quad (2.17)$$

where $J \pm M$ run over integer. The $sl(2)$ -structure has been made transparent by letting

$$f_M^{(J)} \equiv \sqrt{\binom{2J}{J+M}} (f_1)^{J-M} (f_2)^{J+M}, \quad \bar{f}_M^{(J)} \equiv \sqrt{\binom{2J}{J+M}} (\bar{f}_1)^{J+M} (\bar{f}_2)^{J-M}. \quad (2.18)$$

The notation anticipates that $f_M^{(J)}$ and $\bar{f}_M^{(J)}$ form representations of spin J . This is indeed true since f_1, f_2 and \bar{f}_1, \bar{f}_2 span spin $1/2$ representations, by construction. Explicitly one finds

$$I_{\pm} f_M^{(J)} = \sqrt{(J \mp M)(J \pm M + 1)} f_{M \pm 1}^{(J)}, \quad I_3 f_M^{(J)} = M f_M^{(J)}$$

$$\bar{I}_\pm \bar{f}_M^{(J)} = \sqrt{(J \mp M)(J \pm M + 1)} \bar{f}_{M\pm 1}^{(J)}, \quad \bar{I}_3 f_M^{(J)} = M \bar{f}_M^{(J)}, \quad (2.19)$$

where I_ℓ and \bar{I}_ℓ are the infinitesimal generators of the x_+ and x_- components respectively. Moreover, one sees that

$$(I_\ell + \bar{I}_\ell) e^{-2J\sqrt{\gamma}\Phi} = 0 \quad (2.20)$$

so that the exponentials of the Liouville field are group invariants. For Bloch waves, the corresponding chiral fields are noted $V_m^{(J)}$, and $\bar{V}_m^{(J)}$. We shall mostly deal with them in the quantum case. Finally, it is convenient to show how they may be rewritten, at will, in terms of a single free field out of the two we have introduced. Basically, one uses expression of the type 2.8, suitably modified to take account of the periodicity. It is straightforward to verify that one may write

$$V_{-J}^{(J)} = \left(\frac{1}{\sqrt{S'}}\right)^{2J} = e^{2J\sqrt{\gamma}q^{[1]}}, \quad V_J^{(J)} = \left(\frac{S}{\sqrt{S'}}\right)^{2J} = e^{2J\sqrt{\gamma}q^{[2]}}, \quad (2.21)$$

where S may be also rewritten in terms of each free field by the relations

$$S(u) = \left\{ e^{-4\pi p_0^{[1]}\sqrt{\gamma}} \int_0^u V_1^{(-1)}(\rho) d\rho + \int_u^{2\pi} V_1^{(-1)}(\rho) d\rho \right\} \frac{1}{e^{-4\pi p_0^{[1]}\sqrt{\gamma}} - 1} \quad (2.22)$$

$$\frac{-1}{S(u)} = \left\{ e^{-4\pi p_0^{[2]}\sqrt{\gamma}} \int_0^u V_{-1}^{(-1)}(\rho) d\rho + \int_u^{2\pi} V_{-1}^{(-1)}(\rho) d\rho \right\} \frac{1}{e^{-4\pi p_0^{[2]}\sqrt{\gamma}} - 1} \quad (2.23)$$

Then we may rewrite Eq.2.10 as

$$V_m^{(J)} = \sqrt{\binom{2J}{J+m}} V_{-J}^{(J)} S^{J+m} = \sqrt{\binom{2J}{J+m}} V_J^{(J)} (S^{-1})^{J-m} \quad (2.24)$$

This satisfies the periodicity condition, since

$$S(u + 2i\pi) = e^{-4\pi p_0^{[1]}\sqrt{\gamma}} S(u). \quad (2.25)$$

2.2 Quantization

One simply replaces[15] the above Poisson brackets by commutators. Remarkably the above structure carries over to the quantum case. One may show[15], that for generic γ , there exist two equivalent free quantum fields:

$$q^{[j]}(x) = q_0^{[j]} + p_0^{[j]}x + i \sum_{n \neq 0} e^{-inx} p_n^{[j]}/n, \quad p^{[j]}(x) = q^{[j]'}(x). \quad (2.26)$$

such that

$$\left[p^{[1]}(\sigma_1, \tau), p^{[1]}(\sigma_2, \tau)\right] = \left[p^{[2]}(\sigma_1, \tau), p^{[2]}(\sigma_2, \tau)\right] = 2\pi i \delta'(\sigma_1 - \sigma_2),$$

$$p_0^{[1]} = -p_0^{[2]}, \quad (2.27)$$

$$N^{[1]}(p^{[1]})^2 + p_1^{[1]}/\sqrt{\gamma} = N^{[2]}(p^{[2]})^2 + p_1^{[2]}/\sqrt{\gamma} \quad (2.28)$$

$N^{[1]}$ (resp. $N^{[2]}$) denote normal orderings with respect to the modes of $p^{[1]}$ (resp. of $p^{[2]}$). Eq.2.28 defines the stress-energy tensor and the coupling constant γ of the quantum theory. The former generates a representation of the Virasoro algebra with central charge $C = 3 + 1/\gamma$. At an intuitive level, the correspondence between $p^{[1]}$ and $p^{[2]}$ may be understood from the fact that the Verma modules, which they generate, coincide since the highest weights only depend upon $(p_0^{[1]})^2 = (p_0^{[2]})^2$. Next, the quantum version of the differential equation Eq.2.5 is derived as follows. Consider the operator

$$N^{[j]}(e^{\alpha q^{[j]}/2}) \equiv e^{\alpha q_0^{[j]}/2} e^{\alpha p_0^{[j]}x/2} e^{-ix\alpha^2/4} \times \\ \exp\left((\alpha/2)i \sum_{n<0} e^{-inx} p_n^{[j]}/n\right) \exp\left((\alpha/2)i \sum_{n>0} e^{-inx} p_n^{[j]}/n\right), \quad (2.29)$$

where α is a constant to be determined. Clearly one has

$$\frac{d^2}{dx^2} N^{[j]}(e^{\alpha q^{[j]}/2}) = N^{[j]}\left(e^{\alpha q^{[j]}/2} \left(\frac{\alpha^2}{4} p^{[j]2} + \frac{\alpha}{2} p^{[j]'}\right)\right). \quad (2.30)$$

On the other hand, the quantum version of Eqs.2.11 for the Virasoro generators are given by

$$L_n = \frac{1}{2} \left(\sum_r p_r^{[1]} p_{n-r}^{[1]} - in \frac{p_n^{[1]}}{\sqrt{\gamma}} \right) = \left(\sum_r p_r^{[2]} p_{n-r}^{[2]} - in \frac{p_n^{[2]}}{\sqrt{\gamma}} \right), \quad n \neq 0 \\ L_0 = \frac{1}{2} \sum_r p_r^{[1]} p_{-r}^{[1]} + \frac{1}{8\gamma} = \frac{1}{2} \sum_r p_r^{[2]} p_{-r}^{[2]} + \frac{1}{8\gamma}. \quad (2.31)$$

It is straightforward to verify that if α satisfies the equation

$$\alpha + \frac{2}{\alpha} = \frac{1}{\sqrt{\gamma}}, \quad (2.32)$$

then Eq.2.30 may be rewritten as

$$\begin{aligned} \frac{d^2}{dx^2} N^{[j]}(e^{\alpha q^{[j]}/2}) + \frac{\alpha^2}{2} \left(\sum_{n<0} L_n e^{-inx} + \frac{L_0}{2} + \left(\frac{\alpha^2}{32\pi} - \frac{1}{8\gamma} \right) \right) N^{[j]}(e^{\alpha q^{[j]}/2}) + \\ + \frac{\alpha^2}{2} N^{[j]}(e^{\alpha q^{[j]}/2}) \left(\sum_{n>0} L_n e^{-inx} + \frac{L_0}{2} \right) = 0. \end{aligned} \quad (2.33)$$

Since the operators L_n may be equivalently expressed in terms of $p^{[1]}$ or $p^{[2]}$, it follows that $N^{[j]}(\exp(\alpha q^{[j]}/2))$ are solution of the same operator differential equation. Note that, for a given value of γ , Eq.2.32 which is quadratic gives two values of α we shall denote them by α_{\pm} . They are identical to the so-called screening charges of the Coulomb-gas picture. It is convenient to introduce two parametres noted h , and \hat{h} such that

$$h = \pi \alpha_-^2/2, \quad \hat{h} = \pi \alpha_+^2/2. \quad (2.34)$$

They are the quantum-group deformation parameters as we will soon see. Finally, the quantification of the Schrödinger equation gives us four fields noted V_j , and \hat{V}_j :

$$V_j = N^{(j)}(e^{\sqrt{h/2\pi} q^{[j]}}), \quad \hat{V}_j = N^{(j)}(e^{\sqrt{\hat{h}/2\pi} q^{[j]}}), \quad j = 1, 2, \quad (2.35)$$

which obey the differential equations

$$\begin{aligned} -\frac{d^2 V_j(x)}{dx^2} + \left(\frac{h}{\pi} \right) \left(\sum_{n<0} L_n e^{-inx} + \frac{L_0}{2} + \left(\frac{h}{16\pi} - \frac{C-1}{24} \right) \right) V_j(x) \\ + \left(\frac{h}{\pi} \right) V_j(x) \left(\sum_{n>0} L_n e^{-inx} + \frac{L_0}{2} \right) = 0 \end{aligned} \quad (2.36)$$

$$\begin{aligned} -\frac{d^2 \hat{V}_j(x)}{dx^2} + \left(\frac{\hat{h}}{\pi} \right) \left(\sum_{n<0} L_n e^{-inx} + \frac{L_0}{2} + \left(\frac{h}{16\pi} - \frac{C-1}{24} \right) \right) \hat{V}_j(x) \\ + \left(\frac{\hat{h}}{\pi} \right) \hat{V}_j(x) \left(\sum_{n>0} L_n e^{-inx} + \frac{L_0}{2} \right) = 0. \end{aligned} \quad (2.37)$$

These are operator-Schrödinger equations which are the quantum version of Eq.2.5. They are equivalent to the decoupling of Virasoro null-vectors[16].

There are two possible quantum modifications h and \hat{h} , and thus there are four solutions. Since $C = 1 + 3/\gamma$, h , and \hat{h} are given by

$$h = \frac{\pi}{12} \left(C - 13 - \sqrt{(C - 25)(C - 1)} \right), \quad \hat{h} = \frac{\pi}{12} \left(C - 13 + \sqrt{(C - 25)(C - 1)} \right). \quad (2.38)$$

By operator product V_j , $j = 1, 2$, and \hat{V}_j , $j = 1, 2$, generate two infinite families of chiral fields $V_m^{(J)}$, $-J \leq m \leq J$, and $\hat{V}_{\hat{m}}^{(\hat{J})}$, $-\hat{J} \leq \hat{m} \leq \hat{J}$; with $V_{-1/2}^{(1/2)} = V_1$, $V_{1/2}^{(1/2)} = V_2$, and $\hat{V}_{-1/2}^{(1/2)} = \hat{V}_1$, $\hat{V}_{1/2}^{(1/2)} = \hat{V}_2$. The fields $V_m^{(J)}$, $\hat{V}_{\hat{m}}^{(\hat{J})}$, are of the type $(1, 2J + 1)$ and $(2\hat{J} + 1, 1)$, respectively, in the BPZ classification. For the zero-modes, it is simpler[17] to define the rescaled variables

$$\varpi = ip_0^{[1]} \sqrt{\frac{2\pi}{h}}; \quad \widehat{\varpi} = ip_0^{[1]} \sqrt{\frac{2\pi}{\hat{h}}}; \quad \widehat{\varpi} = \varpi \frac{h}{\pi}; \quad \varpi = \widehat{\varpi} \frac{\hat{h}}{\pi}. \quad (2.39)$$

The Hilbert space in which the operators ψ and $\hat{\psi}$ live, is a direct sum[17, 18, 19] of Verma modules $\mathcal{H}(\varpi)$. They are eigenstates of the quasi momentum ϖ , and satisfy $L_n|\varpi\rangle = 0$, $n > 0$; $(L_0 - \Delta(\varpi))|\varpi\rangle = 0$. The corresponding highest weights $\Delta(\varpi)$ may be rewritten as

$$\Delta(\varpi) \equiv \frac{1}{8\gamma} + \frac{(p_0^{[1]})^2}{2} = \frac{h}{4\pi} \left(1 + \frac{\pi}{h} \right)^2 - \frac{h}{4\pi} \varpi^2. \quad (2.40)$$

The commutation relations Eq.2.27 are to be supplemented by the zero-mode ones:

$$[q_0^{[1]}, p_0^{[1]}] = [q_0^{[2]}, p_0^{[2]}] = i. \quad (2.41)$$

It thus follows (see in particular Eq.2.29), that the fields V and \hat{V} shift the quasi momentum $p_0^{[1]} = -p_0^{[2]}$ by a fixed amount. For an arbitrary c-number function f , one has

$$V_m^{(J)} f(\varpi) = f(\varpi + 2m) V_m^{(J)}, \quad \hat{V}_{\hat{m}}^{(\hat{J})} f(\varpi) = f(\varpi + 2\hat{m} \pi/h) \hat{V}_{\hat{m}}^{(\hat{J})}. \quad (2.42)$$

The fields V and \hat{V} together with their products noted $V_{m\hat{m}}^{(J\hat{J})}$ may thus be naturally restricted to spaces with discrete values of ϖ .

3 The OPA of the $U_q(sl(2))$ family.

In this section, as well as in the next one we only consider for simplicity only the case of the operators $V_m^{(J)} \equiv V_{m0}^{(J0)}$, which involve a single screening charge.

3.1 The fusing and braiding matrices of the V fields

The complete operator algebra of the V fields was spelled out recently[9] and put in correspondence with the general scheme of Moore and Seiberg[14]. We shall first describe the result and give a summary of the derivation later on. The Moore Seiberg (MS) chiral vertex-operators connect three specified Verma modules and are thus of the form $\phi_{J_3, J_1}^{J_2}$. According to Eq.2.42, the $V_m^{(J)}$ operators, on the contrary, naturally act in Hilbert spaces of the form

$$\mathcal{H} \equiv \bigoplus_{n=-\infty}^{+\infty} \mathcal{H}(\varpi_0 + n). \quad (3.1)$$

where $\mathcal{H}(\varpi_0 + n)$ are Verma modules. ϖ_0 is a constant. We shall choose $\varpi_0 = 1 + \pi/h$, which corresponds to the $sl(2, C)$ -invariant vacuum. With this choice, we use the notation \mathcal{H}_J instead of $\mathcal{H}(\varpi_0 + 2J)$. It is now easy to see that, according to Eq.2.42, the V fields and the MS fields are related by the projection operator \mathcal{P}_J :

$$\mathcal{P}_J \mathcal{H} = \mathcal{H}_J, \quad \mathcal{P}_{J_3} V_m^{(J_2)} \equiv \phi_{J_3, J_3+m}^{J_2} \quad (3.2)$$

The V fields are such that $\langle \varpi_2 | V_m^{(J)} | \varpi_1 \rangle$ is equal to one if $\varpi_1 = \varpi_3 + 2m$, and is equal to zero otherwise. This normalization is required by the symmetry between three legs (sphere with three punctures). From now on, we restrict ourselves to the case of genus zero, so that we perform a conformal transformation on the operator to change from the coordinate x to the coordinate $z = \exp(ix)$. The complete fusion and braiding algebras take the form:

$$\begin{aligned} \mathcal{P}_K V_{m_1}^{(J_1)}(z_1) V_{m_2}^{(J_2)}(z_2) &= \sum_{J=|J_1-J_2|}^{J_1+J_2} F_{K+m_1, J} \begin{bmatrix} J_1 & J_2 \\ K & K+m_1+m_2 \end{bmatrix} \\ \sum_{\{\nu\}} \mathcal{P}_K V_{m_1+m_2}^{(J, \{\nu\})}(z_2) &\langle \varpi_J, \{\nu\} | V_{J_2-J}^{(J_1)}(z_1 - z_2) | \varpi_{J_2} \rangle; \end{aligned} \quad (3.3)$$

$$\mathcal{P}_K V_{m_1}^{(J_1)}(z_1) V_{m_2}^{(J_2)}(z_2) = \sum_{n_2} B_{K+m_1, K+n_2} \begin{bmatrix} J_1 & J_2 \\ K & K+m_1+m_2 \end{bmatrix} \mathcal{P}_K V_{n_2}^{(J_2)}(z_2) V_{m_1+m_2-n_2}^{(J_1)}(z_1) \quad (3.4)$$

where F and B (the fusing and braiding matrices) have numerical entries. The notation $|\varpi_J, \{\nu\}\rangle$ represents an arbitrary state in \mathcal{H}_J . Using Eq.3.2, one may verify that these expressions have the general MS form. In ref.[9], it was shown that

$$F_{J_{23}, J_{12}} \begin{bmatrix} J_1 & J_2 \\ J_{123} & J_3 \end{bmatrix} = \frac{g_{J_1 J_2}^{J_{12}} g_{J_{12} J_3}^{J_{123}}}{g_{J_2 J_3}^{J_{23}} g_{J_1 J_{23}}^{J_{123}}} \left\{ \begin{matrix} J_1 & J_2 & J_{12} \\ J_3 & J_{123} & J_{23} \end{matrix} \right\}_q. \quad (3.5)$$

The symbol $\left\{ \begin{matrix} J_1 & J_2 & J_{12} \\ J_3 & J & J_{23} \end{matrix} \right\}_q$ represents the quantum 6-j coefficient which is not completely tetrahedron-symmetric, as the notation indicates. This is in contrast with the Racah-Wigner q-6-j symbol, noted $\left\{ \begin{matrix} J_1 & J_2 & J_{12} \\ J_3 & J & J_{23} \end{matrix} \right\}_q$. Recall that the q-6-j symbols are quantum-group recoupling coefficients which satisfy the defining relation

$$\sum_{J_{23}} (J_2, M_2; J_3, M_3 | J_{23})_q (J_1, M_1; J_{23}, M_{23} | J_{123})_q \left\{ \begin{matrix} J_1 & J_2 & J_{12} \\ J_3 & J_{123} & J_{23} \end{matrix} \right\}_q = (J_1, M_1; J_2, M_2 | J_{12})_q (J_{12}, M_{12}; J_3, M_3 | J_{123})_q. \quad (3.6)$$

The symbols $(J_k, M_k; J_\ell, M_\ell | J_{k\ell})_q$ denote the q Clebsch-Gordan (3-j) coefficients³.

This term was of course expected, in view of the quantum-group structure previously exhibited, in particular, in refs.[17, 19]. However, there appear, in addition, coupling constants $g_{J_1 J_2}^{J_{12}}$, which are not trigonometrical functions of h . Their general expression is

$$g_{J_1 J_2}^{J_{12}} = \left(\frac{h}{\pi} \right)^{J_1+J_2-J_{12}} \prod_{k=1}^{J_1+J_2-J_{12}} \sqrt{\frac{F(1 + (2J_1 - k + 1)h/\pi)}{F(1 + kh/\pi)}} \sqrt{F(1 + (2J_2 - k + 1)h/\pi)F(-1 - (2J_{12} + k + 1)h/\pi)}. \quad (3.7)$$

where $F(z) = \Gamma(z)/\Gamma(1-z)$. Note that $\Gamma(z)$ is the standard — not q-deformed — Gamma function. The result is symmetric in J_1, J_2 . The lack

³We use a condensed notation $(J_1, M_1; J_2, M_2 | J_{12})_q$ instead of $(J_1, M_1; J_2, M_2 | J_{12}, M_1 + M_2)_q$.

of symmetry between J_1 or J_2 and J_{12} is due to the particular metric in the space of primary fields. The F and B matrices are found to be connected by the MS relation[14]

$$B_{J_{23}, J_{12}}^{\pm} \begin{bmatrix} J_1 & J_3 \\ J & J_2 \end{bmatrix} = e^{\pm i\pi(\Delta_J + \Delta_{J_2} - \Delta_{J_{23}} - \Delta_{J_{12}})} F_{J_{23}, J_{12}} \begin{bmatrix} J_1 & J_2 \\ J & J_3 \end{bmatrix}. \quad (3.8)$$

where Δ_J is the conformal weight

$$\Delta_J = -\frac{h}{\pi} J(J+1) - J \quad (3.9)$$

The symbol \pm is chosen according to the ordering of the operator: taking for instance, $z_i = \exp(i\sigma_i)$, $0 \leq \sigma_i \leq \pi$, one has $\pm = -\text{sign}(\sigma_1 - \sigma_2)$. The explicit formula for the Racah-Wigner q-6-j coefficients, which have the tetrahedral symmetry, are given in [20] by

$$\begin{aligned} \left\{ \begin{matrix} a & b & e \\ d & c & f \end{matrix} \right\}_q &= ([2e+1][2f+1])^{-1/2} (-1)^{a+b-c-d-2e} \left\{ \begin{matrix} a & b & e \\ d & c & f \end{matrix} \right\}_q = \\ &= \Delta(a, b, e) \Delta(a, c, f) \Delta(c, e, d) \Delta(d, b, f) \times \\ &\sum_{z \text{ integer}} (-1)^z [z+1]! \left[[z-a-b-e]! [z-a-c-f]! [z-b-d-f]! \times \right. \\ &\left. [z-d-c-e]! [a+b+c+d-z]! [a+d+e+f-z]! [b+c+e+f-z]! \right]^{-1} \end{aligned} \quad (3.10)$$

with

$$\Delta(l, j, k) = \sqrt{\frac{[-l+j+k]! [l-j+k]! [l+j-k]!}{[l+j+k+1]!}}$$

and

$$[n]! \equiv \prod_{r=1}^n [r] \quad [r] \equiv \frac{\sin(hr)}{\sin h}. \quad (3.11)$$

In the operator formalism of the Liouville theory, the fusion may be rewritten as⁴

$$V_{m_1}^{(J_1)} V_{m_2}^{(J_2)} = \sum_{J_{12}=|J_1-J_2|}^{J_1+J_2} \frac{g_{J_1 J_2}^{J_{12}} g_{J_{12} (\varpi - \varpi_0 + 2m_1 + 2m_2)/2}^{(\varpi - \varpi_0)/2}}{g_{J_2 (\varpi - \varpi_0 + 2m_1 + 2m_2)/2}^{(\varpi - \varpi_0 + 2m_1)/2} g_{J_1 (\varpi - \varpi_0 + 2m_1)/2}^{(\varpi - \varpi_0)/2}} \times$$

⁴We omit the world-sheet variables from now on.

$$\left\{ \begin{matrix} J_1 & J_2 \\ (\varpi - \varpi_0 + 2m_1 + 2m_2)/2 & (\varpi - \varpi_0)/2 \end{matrix} \middle| \begin{matrix} J_{12} \\ (\varpi - \varpi_0 + 2m_1)/2 \end{matrix} \right\}_q \times \sum_{\{\nu_{12}\}} V_{m_1+m_2}^{(J_{12}, \{\nu_{12}\})} < \varpi_{J_{12}}, \{\nu_{12}\} | V_{J_2-J_{12}}^{(J_1)} | \varpi_{J_2} >. \quad (3.12)$$

In this last formula, ϖ is an operator. It is easy to check that this operator-expression is equivalent to Eq.3.3, by computing the matrix element between the states $< \varpi_{J_{123}}, \{\nu_{123}\} |$, and $| \varpi_{J_3}, \{\nu_3\} >$. Then, the additional spins of Eq.3.3, as compared with Eq.3.12, are given by

$$\begin{aligned} J_{123} &= (\varpi - \varpi_0)/2, \\ J_{23} &= (\varpi - \varpi_0 + 2m_1)/2, \\ J_3 &= (\varpi - \varpi_0 + 2m_1 + 2m_2)/2. \end{aligned} \quad (3.13)$$

Similarly, the braiding may be written as

$$\begin{aligned} V_{m_1}^{(J_1)} V_{m_2}^{(J_2)} &= \sum_{n_1+n_2=m_1+m_2} e^{\mp i h (2m_1 m_2 + m_2^2 - n_2^2 + \varpi(m_2 - n_2))} \times \\ &\left\{ \begin{matrix} J_1 & (\varpi - \varpi_0 + 2m_1 + 2m_2)/2 \\ J_2 & (\varpi - \varpi_0)/2 \end{matrix} \middle| \begin{matrix} (\varpi - \varpi_0 + 2n_2)/2 \\ (\varpi - \varpi_0 + 2m_1)/2 \end{matrix} \right\} \times \\ &\frac{g_{J_1}^{(\varpi - \varpi_0 + 2n_2)/2} g_{J_2}^{(\varpi - \varpi_0)/2}}{g_{J_1}^{(\varpi - \varpi_0 + 2m_1)/2} g_{J_2}^{(\varpi - \varpi_0 + 2m_2)/2}} V_{n_2}^{(J_2)} V_{n_1}^{(J_1)}. \end{aligned} \quad (3.14)$$

The correspondence table is again given by Eq.3.13 with, in addition,

$$J_{13} = (\varpi - \varpi_0 + 2n_2)/2. \quad (3.15)$$

In the operator-forms Eqs.3.12, 3.14, one sees that the fusion and braiding matrices involve the operator ϖ , and thus do not commute with the V-operators (see Eq.2.42). Such is the general situation of the operator-algebras in the MS formalism. This is in contrast with, for instance, the braiding relations for quantum group representations. In the article[10], and completing the results of refs.[4, 17], it was shown how to change basis to the holomorphic operators ξ which are such that these ϖ dependences of the fusing and braiding matrices disappear. After the transformation, the fusing and braiding matrices become equal to the q-Clebsch-Gordan coefficients and to the universal R matrix, respectively; and the quantum group structure becomes more transparent. We shall not elaborate on this point here.

3.2 The polynomial equations

They basically express the associativity of the operator product algebra (OPA). First, consider the fusion. The associated pentagonal equation is derived as follows. We fuse $\langle \varpi_J | V_{m_1}^{(J_1)} V_{m_2}^{(J_2)} V_{m_3}^{(J_3)} | \varpi_{J+2m_1+2m_2+2m_3} \rangle$ in two different ways, beginning from the left, and from the right, and identify the coefficients of the resulting operator $V_{m_1+m_2+m_3}^{(J_{123}, \{\nu_{123}\})}$. This gives

$$\begin{aligned}
& \sum_{J_{12}} F_{J+m_1+m_2, J_{123}} \left[\begin{matrix} J_{12} & J_3 \\ J & J+m_1+m_2+m_3 \end{matrix} \right] F_{J+m_1, J_{12}} \left[\begin{matrix} J_1 & J_2 \\ J & J+m_1+m_2 \end{matrix} \right] \times \\
& \sum_{\{\nu_{12}\}} \langle \varpi_{J_{123}}, \{\nu_{123}\} | V_{J_3-J_{123}}^{J_{12}, \{\nu_{12}\}} | \varpi_{J_3} \rangle \langle \varpi_{J_{12}}, \{\nu_{12}\} | V_{J_2-J_{12}}^{J_1} | \varpi_{J_2} \rangle = \\
& \sum_{J_{23}} F_{J+m_1, J_{123}} \left[\begin{matrix} J_1 & J_{23} \\ J & J+m_1+m_2+m_3 \end{matrix} \right] F_{J+m_1+m_2, J_{23}} \left[\begin{matrix} J_2 & J_3 \\ J+m_1 & J+m_1+m_2+m_3 \end{matrix} \right] \times \\
& \sum_{\{\nu_{23}\}} \langle \varpi_{J_{123}}, \{\nu_{123}\} | V_{J_{23}-J_{123}}^{J_1} | \varpi_{J_{23}}, \{\nu_{23}\} \rangle \langle \varpi_{J_{23}}, \{\nu_{23}\} | V_{J_3-J_{23}}^{J_2} | \varpi_{J_3} \rangle .
\end{aligned} \tag{3.16}$$

On the r.h.s. we use

$$\sum_{\{\nu_{23}\}} | \varpi_{J_{23}}, \{\nu_{23}\} \rangle \langle \varpi_{J_{23}}, \{\nu_{23}\} | = \mathcal{P}_{J_{23}}, \tag{3.17}$$

and obtain the factor $\langle \varpi_{J_{123}}, \{\nu_{123}\} | V_{J_{23}-J_{123}}^{J_1} V_{J_3-J_{23}}^{J_2} | \varpi_{J_3} \rangle$. These last operators are fused in their turn, obtaining

$$\begin{aligned}
& \sum_{J_{23}} F_{J+m_1+m_2, J_{23}} \left[\begin{matrix} J_2 & J_3 \\ J+m_1 & J+m_1+m_2+m_3 \end{matrix} \right] F_{J+m_1, J_{123}} \left[\begin{matrix} J_1 & J_{23} \\ J & J+m_1+m_2+m_3 \end{matrix} \right] F_{J_{23}, J_{12}} \left[\begin{matrix} J_1 & J_2 \\ J_{123} & J_3 \end{matrix} \right] \\
& = F_{J+m_1, J_{12}} \left[\begin{matrix} J_1 & J_2 \\ J & J+m_1+m_2 \end{matrix} \right] F_{J+m_1+m_2, J_{123}} \left[\begin{matrix} J_{12} & J_3 \\ J & J+m_1+m_2+m_3 \end{matrix} \right].
\end{aligned} \tag{3.18}$$

On the other hand, the q-6j symbols are well known to satisfy the same relation[20], that is

$$\begin{aligned}
& \sum_{J_{23}} \left\{ \begin{matrix} J_2 & J_3 \\ J+m_1+m_2+m_3 & J+m_1 \end{matrix} \middle| \begin{matrix} J_{23} \\ J+m_1+m_2 \end{matrix} \right\}_q \left\{ \begin{matrix} J_1 & J_{23} \\ J+m_1+m_2+m_3 & J \end{matrix} \middle| \begin{matrix} J_{123} \\ J+m_1 \end{matrix} \right\}_q \left\{ \begin{matrix} J_1 & J_2 \\ J_3 & J_{123} \end{matrix} \middle| \begin{matrix} J_{12} \\ J_{23} \end{matrix} \right\}_q \\
& = \left\{ \begin{matrix} J_1 & J_2 \\ J+m_1+m_2 & J \end{matrix} \middle| \begin{matrix} J_{12} \\ J+m_1 \end{matrix} \right\}_q \left\{ \begin{matrix} J_{12} & J_3 \\ J+m_1+m_2+m_3 & J \end{matrix} \middle| \begin{matrix} J_{123} \\ J+m_1+m_2 \end{matrix} \right\}_q.
\end{aligned} \tag{3.19}$$

Another example is the Yang Baxter equation

$$\begin{aligned} & \sum_{J_{134}} B_{J_{234}, J_{134}} \left[\begin{matrix} J_1 & J_2 \\ J_{1234} & J_{34} \end{matrix} \right] B_{J_{34}, J_{14}} \left[\begin{matrix} J_1 & J_3 \\ J_{134} & J_4 \end{matrix} \right] B_{J_{134}, J_{124}} \left[\begin{matrix} J_2 & J_3 \\ J_{1234} & J_{14} \end{matrix} \right] \\ &= \sum_{J_{24}} B_{J_{34}, J_{24}} \left[\begin{matrix} J_2 & J_3 \\ J_{234} & J_4 \end{matrix} \right] B_{J_{234}, J_{124}} \left[\begin{matrix} J_1 & J_3 \\ J_{1234} & J_{24} \end{matrix} \right] B_{J_{24}, J_{14}} \left[\begin{matrix} J_1 & J_2 \\ J_{124} & J_4 \end{matrix} \right], \end{aligned} \quad (3.20)$$

which is obtained from operator-braidings in $\langle \varpi_{J_{1234}} | V^{(J_1)} V^{(J_2)} V^{(J_3)} | \varpi_{J_4} \rangle$. The 6-j symbols also satisfy a similar equation

$$\begin{aligned} & \sum_{J_{134}} e^{i\pi\epsilon(\Delta_{J_{1234}} - \Delta_{J_{124}} - \Delta_{J_{134}} - \Delta_{J_{234}})} \left\{ \begin{matrix} J_1 & J_{34} \\ J_2 & J_{1234} \end{matrix} \middle| \begin{matrix} J_{134} \\ J_{234} \end{matrix} \right\}_q \left\{ \begin{matrix} J_1 & J_4 \\ J_3 & J_{134} \end{matrix} \middle| \begin{matrix} J_{14} \\ J_{34} \end{matrix} \right\}_q \left\{ \begin{matrix} J_2 & J_{14} \\ J_3 & J_{1234} \end{matrix} \middle| \begin{matrix} J_{124} \\ J_{134} \end{matrix} \right\}_q \\ &= \sum_{J_{24}} e^{i\pi\epsilon(\Delta_{J_4} - \Delta_{J_{14}} - \Delta_{J_{24}} - \Delta_{J_{34}})} \left\{ \begin{matrix} J_2 & J_4 \\ J_3 & J_{234} \end{matrix} \middle| \begin{matrix} J_{24} \\ J_{34} \end{matrix} \right\}_q \left\{ \begin{matrix} J_1 & J_{24} \\ J_3 & J_{1234} \end{matrix} \middle| \begin{matrix} J_{124} \\ J_{234} \end{matrix} \right\}_q \left\{ \begin{matrix} J_1 & J_4 \\ J_2 & J_{124} \end{matrix} \middle| \begin{matrix} J_{14} \\ J_{24} \end{matrix} \right\}_q, \end{aligned} \quad (3.21)$$

Quite generally, the 6-j symbols are solutions of all polynomial equations. It is quite simple to see that the gg/gg factors are “pure gauges ” from the viewpoint of MS coditions since they cancel in pairs. Indeed one may define new chiral vertex operators \tilde{V} , by

$$\mathcal{P}_{J_{12}} \tilde{V}_{J_2 - J_{12}}^{(J_1)} \equiv g_{J_1 J_2}^{J_{12}} \mathcal{P}_{J_{12}} V_{J_2 - J_{12}}^{(J_1)}, \quad (3.22)$$

so that the fusing and braiding matrices simply become

$$\begin{aligned} & \tilde{F}_{J_{23}, J_{12}} \left[\begin{matrix} J_1 & J_2 \\ J & J_3 \end{matrix} \right] = \left\{ \begin{matrix} J_1 & J_2 \\ J_3 & J \end{matrix} \middle| \begin{matrix} J_{12} \\ J_{23} \end{matrix} \right\}_q, \\ & \tilde{B}_{J_{23}, J_{12}}^{\pm} \left[\begin{matrix} J_1 & J_3 \\ J & J_2 \end{matrix} \right] = e^{\pm i\pi(\Delta_J + \Delta_{J_2} - \Delta_{J_{23}} - \Delta_{J_{12}})} \left\{ \begin{matrix} J_1 & J_2 \\ J_3 & J \end{matrix} \middle| \begin{matrix} J_{12} \\ J_{23} \end{matrix} \right\}_q. \end{aligned} \quad (3.23)$$

In the associativity conditions for the \tilde{V} operators the g ’s have disappeared. Let us stress, however, as already mentioned in the introduction, that **this does not mean that our fusing and braiding matrices are equivalent to Eqs.3.23 from the viewpoint of conformal theory**. Indeed, the new fusing equations read

$$\begin{aligned} & \mathcal{P}_J \tilde{V}_{m_1}^{(J_1)} \tilde{V}_{m_2}^{(J_2)} = \mathcal{P}_J \sum_{J_{12}=|J_1-J_2|}^{J_1+J_2} \tilde{F}_{J+m_1, J_{12}} \left[\begin{matrix} J_1 & J_2 \\ J & J+m_1+m_2 \end{matrix} \right] \times \\ & \sum_{\{\nu_{12}\}} \tilde{V}_{m_1+m_2}^{(J_{12}, \{\nu_{12}\})} < \varpi_{J_{12}}, \{\nu_{12}\} | \tilde{V}_{J_2 - J_{12}}^{(J_1)} | \varpi_{J_2} >. \end{aligned} \quad (3.24)$$

The matrix element of \tilde{V} on the right-hand side is a book-keeping device to recover the coefficients of the OPE. There the normalization of \tilde{V} appears explicitly. According to Eqs.3.22, the matrix elements of \tilde{V} are such that

$$\langle \varpi_{J_2} | \tilde{V}_m^{(J_1)}(1) | \varpi_{J_{12}} \rangle = g_{J_1 J_2}^{J_{12}} \delta_{m, J_{12}-J_2}. \quad (3.25)$$

For instance, at the level of primaries we have

$$\mathcal{P}_J \tilde{V}_{m_1}^{(J_1)} \tilde{V}_{m_2}^{(J_2)} = \mathcal{P}_J \sum_{J_{12}=|J_1-J_2|}^{J_1+J_2} g_{J_1 J_2}^{J_{12}} \tilde{F}_{J+m_1, J_{12}} \begin{bmatrix} J_1 & J_2 \\ J & J+m_1+m_2 \end{bmatrix} \tilde{V}_{m_1+m_2}^{(J_{12})} + \dots \quad (3.26)$$

The g 's have re-appeared. There is no way to get rid of them at the level of the two dimensional OPA.

3.3 Some ideas about the derivation of the OPA

It follows from the operator differential-equation Eq.2.36 that, for any holomorphic primary operator $A_\Delta(x)$ with conformal weight Δ , one has (see Eq.A.6 of ref[17]):

$$\begin{aligned} \langle \varpi_4 | V_{\pm 1/2}^{(1/2)}(x) A_\Delta(x') | \varpi_1 \rangle &= e^{ix'(\varpi_1^2 - \varpi_4^2)h/4\pi} e^{i(x'-x)(-1/2 \mp \varpi_4)h/2\pi} \\ &\times (1 - e^{-i(x-x')})^\beta F(a_\pm, b_\pm; c_\pm; e^{i(x'-x)}), \end{aligned} \quad (3.27)$$

$$\begin{aligned} a_\pm &= \beta - \frac{h}{2\pi} \mp h \left(\frac{\varpi_4 - \varpi_1}{2\pi} \right); \quad b_\pm = \beta - \frac{h}{2\pi} \mp h \left(\frac{\varpi_4 + \varpi_1}{2\pi} \right); \quad c_\pm = 1 \mp \frac{h\varpi_4}{\pi}; \\ \beta &= \frac{1}{2}(1 + h/\pi) \left(1 - \sqrt{1 - \frac{8h\Delta}{2\pi(1 + h/\pi)^2}} \right); \end{aligned} \quad (3.28)$$

where $F(a, b; c; z)$ is the standard hypergeometric function. Moreover

$$\begin{aligned} \langle \varpi_4 | A_\Delta(x') V_{\pm 1/2}^{(1/2)}(x) | \varpi_1 \rangle &= e^{ix'(\varpi_1^2 - \varpi_4^2)h/4\pi} e^{i(x-x')(-1/2 \pm \varpi_1)h/2\pi} \times \\ &(1 - e^{-i(x-x')})^\beta F(a'_\pm, b'_\pm; c'_\pm; e^{-i(x-x')}), \end{aligned} \quad (3.29)$$

$$a'_\pm = \beta - \frac{h}{2\pi} \mp h \left(\frac{\varpi_1 - \varpi_4}{2\pi} \right); \quad b'_\pm = \beta - \frac{h}{2\pi} \mp h \left(\frac{\varpi_1 + \varpi_4}{2\pi} \right); \quad c'_\pm = 1 \mp \frac{h\varpi_1}{\pi}. \quad (3.30)$$

Consider the fusion. Making use of the well known identity

$$F(a, b; c; x) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F(a, b; a+b-c+1; 1-x) + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-x)^{c-a-b} F(c-a, c-b; c-a-b+1; 1-x), \quad (3.31)$$

one derives Eq.3.3, with $J_1 = 1/2$, and

$$F_{J+\epsilon_1/2, J_2+\epsilon_2/2} \begin{bmatrix} 1/2 & J_2 \\ J & J_3 \end{bmatrix} = \frac{\Gamma((1-\epsilon_1) - \epsilon_1(2J+1)h/\pi)}{\Gamma(1 + (\epsilon_2 - \epsilon_1)/2 + (J_3 + \epsilon_2 J_2 - \epsilon_1 J + (1 - \epsilon_1 + \epsilon_2)/2)h/\pi)} \times \frac{\Gamma(\epsilon_2 + \epsilon_2(2J_2+1)h/\pi)}{\Gamma((\epsilon_2 - \epsilon_1)/2 + (-J_3 + \epsilon_2 J_2 - \epsilon_1 J + (-1 - \epsilon_1 + \epsilon_2)/2)h/\pi)}, \quad (3.32)$$

where $\epsilon_i = \pm 1$. Introduce

$$A \begin{bmatrix} J_1 & J_2 & J_{12} \\ J_3 & J & J_{23} \end{bmatrix} \equiv \frac{F_{J_{23}, J_{12}} \begin{bmatrix} J_1 & J_2 \\ J & J_3 \end{bmatrix}}{\left\{ \begin{matrix} J_1 & J_2 \\ J_3 & J \end{matrix} \middle| \begin{matrix} J_{12} \\ J_{23} \end{matrix} \right\}_q}. \quad (3.33)$$

Eq.3.32 shows that

$$A \begin{bmatrix} 1/2 & J_2 & J_2+\epsilon_2/2 \\ J_3 & J & J+\epsilon_1/2 \end{bmatrix} = \sqrt{\frac{F(\epsilon_2 + \epsilon_2(2J_2+1)h/\pi)}{F((1 + (\epsilon_2 - \epsilon_1)/2) + (\epsilon_2 J_2 - \epsilon_1 J + J_3 + (1 - \epsilon_1 + \epsilon_2)/2)h/\pi)}} \times \sqrt{\frac{F((1 - \epsilon_1) - \epsilon_1(2J+1)h/\pi)}{F((\epsilon_2 - \epsilon_1)/2 + (\epsilon_2 J_2 - \epsilon_1 J - J_3 + (-1 - \epsilon_1 + \epsilon_2)/2)h/\pi)}}, \quad (3.34)$$

Assume next that the fusing matrix takes the form Eq.3.5. Using the explicit expression Eq.3.10, this gives

$$\frac{g_{J_2+\epsilon_2/2, J_3}^J}{g_{J_2 J_3}^{J+\epsilon_1/2}} = \frac{g_{1/2, J+\epsilon_1/2}^J}{g_{J_2+\epsilon_2/2}^{J_2+\epsilon_2/2}} A \begin{bmatrix} 1/2 & J_2 & J_2+\epsilon_2/2 \\ J_3 & J & J+\epsilon_1/2 \end{bmatrix}. \quad (3.35)$$

Clearly, once we know the expression of $g_{1/2 J}^{J\pm 1/2}$, this last relation will allow us to determine all of the $g_{J_1 J_2}^{J_{12}}$ by a double recursion on the indices J_1 and

J_{12} (for any J_2). The basic point is that the integrability condition for this recurrence relation, completely fixes $g_{1/2 J}^{J \pm 1/2}$. Note that $g_{1/2 J}^{J+1/2}$ can be taken equal to one, without loss of generality. For, if it were not, it would be possible to define

$$\tilde{g}_{J_1 J_2}^{J_{12}} \equiv \frac{\alpha_{J_1} \alpha_{J_2}}{\alpha_{J_{12}}} g_{J_1 J_2}^{J_{12}} \quad \text{such that} \quad \tilde{g}_{1/2 J}^{J+1/2} \equiv \frac{\alpha_{1/2} \alpha_J}{\alpha_{J+1/2}} g_{1/2 J}^{J+1/2} = 1,$$

and the fusion coefficients F defined by Eq.3.5 with \tilde{g} are the same as the ones defined with g . With this condition, the integrability condition gives, up to an irrelevant constant factor (it does not affect F):

$$g_{1/2 J}^{J-1/2} = g_0 \sqrt{F(1 + 2Jh/\pi)F(-1 - (2J+1)h/\pi)}. \quad (3.36)$$

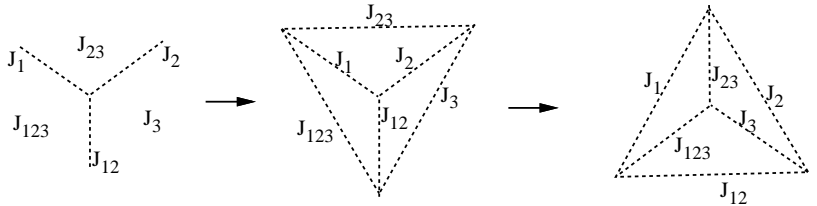
Finally, one solves the recurrence relations, and compute the general expression for $g_{J_1 J_2}^{J_{12}}$, thereby deriving Eq.3.7.

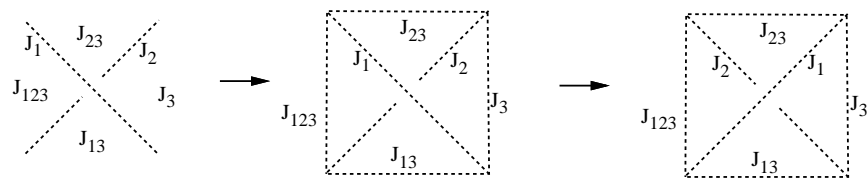
This completes the derivation of Eq.3.5 for $J_1 = 1/2$. The final part of the proof is to use Eqs.3.18, and 3.19 to set up a recurrence on J_1 , starting from our previous derivation of the case $J_1 = 1/2$. Assume that Eq.3.5 holds for $J_1 \leq K$, and write Eqs.3.18, and 3.19 for $J_1 \leq K$ and $J_2 \leq K$. In Eq.3.18, the last term, that is $F_{J+m_1+m_2, J_{123}} \left[\begin{smallmatrix} J_{12} & J_3 \\ J & J+m_1+m_2+m_3 \end{smallmatrix} \right]$ may have $J_{12} > K$, and then it is the only one which is not known from our hypothesis. Combining this equality with the similar one for 6-j symbols one immediately sees that $F_{J+m_1+m_2, J_{123}} \left[\begin{smallmatrix} J_{12} & J_3 \\ J & J+m_1+m_2+m_3 \end{smallmatrix} \right]$ is also given by Eq.3.5. Thus this relation holds for all J 's.

3.4 The general (3D) structure

In this section we discuss the general structure of the bootstrap equations. In ref.[20], quantum-group diagrams were introduced which involve two different “worlds”: the “normal” one and the “shadow” one. Adopting this terminology from now on, we are going to verify that the OPA of the V is in exact correspondence with the shadow diagrams. At the same time we shall discuss the associated three dimensional aspect. It corresponds to the quantum-group version of the Regge-calculus approach to the discrete three-dimensional gravity[21] or to the discussion of ref.[22], for instance. In the pictorial representations, we omit the g coefficients. Thus, we actually make use of the operator-algebra expressed in terms of the \tilde{V} fields. Though of

great importance for operator-product expansion, the coupling constants g define a pure gauge for the polynomial equations or knot-theory viewpoints. We could draw other figures including g coefficients, to show how they cancel out of those equations, but this would be cumbersome. The basic fusing and braiding operations on the \tilde{V} operators have three equivalent representations



$$= \left\{ \begin{matrix} J_1 & J_2 \\ J_3 & J_{123} \end{matrix} \middle| \begin{matrix} J_{12} \\ J_{23} \end{matrix} \right\} \text{ (fusion of } \tilde{V} \text{ operators),} \quad (3.37)$$


$$= e^{i\pi(\Delta_{J_{123}} + \Delta_{J_3} - \Delta_{J_{23}} - \Delta_{J_{13}})} \left\{ \begin{matrix} J_1 & J_3 \\ J_2 & J_{123} \end{matrix} \middle| \begin{matrix} J_{13} \\ J_{23} \end{matrix} \right\} \text{ (braiding of } \tilde{V} \text{ operators).} \quad (3.38)$$

First consider the left diagrams, and the associated Eqs.3.3, and 3.4. Apart from the \tilde{V} -matrix element on the right-hand side of the fusing relation, which has no specific representative, each operator $\tilde{V}_m^{(J)}$ is represented by a dashed line carrying the label J . The spins on the faces display the zero-modes of the Verma modules on which the $\tilde{V}_m^{(J)}$ operators act. Thus the m 's are differences between the spin-labels of the two neighbouring faces. For the braiding diagram, the spins on the edges are unchanged at crossings, and, for given J_1, J_2 , the braiding diagram has the form of a vertex of an interaction-around-the-face (IRF) model. These diagrams are two-dimensional (2D). The appearance of spins on the faces reflect the fact that the fusion and braiding properties depend upon the Verma module on which the operators act. It is easily seen that, when they are used as building blocks, the above drawings generate diagrams which have the same structure as the quantum-group ones of ref.[20] in the shadow world⁵. The polynomial equations can

⁵we used dashed lines to agree with the conventions of ref.[20].

be viewed as link-invariance conditions. For instance,

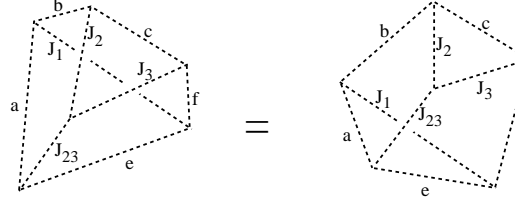
$$(3.39)$$

gives the pentagonal relation of the V fields discussed in ref.[9], after cancellation of the phases (with a change of indices).

The middle diagrams of figures 3.37, and 3.38 are obtained from the left ones (first arrow) by enclosing the 2D figures with extra dashed lines carrying the spin labels which were previously on the faces. In this way, one gets three-dimensional (3D) tetrahedra, with spin labels only on the edges. The right figures are obtained from the middle ones by dualisation: the face, surrounded by the edges J_a, J_b, J_c , becomes the vertex where the edges J_a, J_b, J_c join, and conversely, a vertex becomes a face. An edge joining two vertices becomes the edge between the two dual faces. There is one triangular face for each \tilde{V} field, including the \tilde{V} matrix-element of the fusing relation Eq.3.3. On the dualised polyhedra, the triangular inequalities give the addition rules for spins. The main point of the middle and right diagrams is that, as a consequence of the basic MS properties of the OPA, they are really 2D projections of three-dimensional diagrams which may be rotated at essentially no cost⁶. For instance, the MS relation between fusing and braiding matrices simply corresponds to the fact that they are represented by tetrahedra which may be identified after a rigid 3D rotation. We shall illustrate the general properties of the 3D diagrams on the example of the pentagonal relation. In the same way as we closed the basic figures in Eq.3.37, 3.38, the rule to go to 3D is to close the composite figure 3.39. It gives a polyhedron which has vertices with three edges only, which we call type V3E. The two-dimensional Eq.3.39 now simply corresponds to viewing the

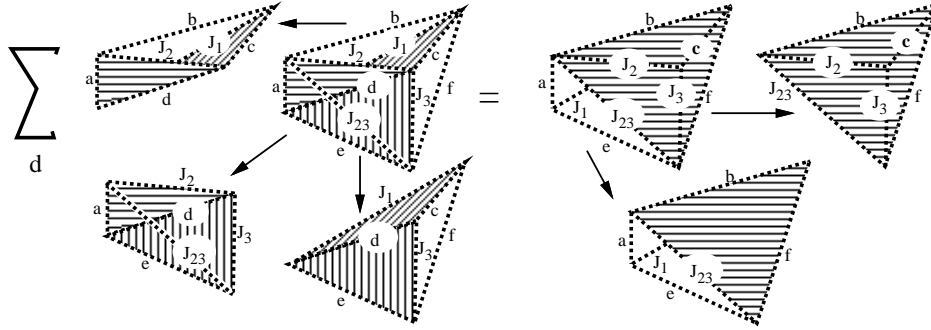
⁶We use 6-j symbols which do not have the full tetrahedral symmetry, so that two edges should be distinguished. This is discussed in ref.[9].

V3E polyhedron from two different angles:



$$(3.40)$$

Dualisation gives a polyhedron, with only triangular faces, which we call F3E. The polynomial equations are recovered by decomposing a F3E polyhedron into tetrahedra (this correspondence only works with F3E polyhedra, this is the reason for dualisation). In parallel with the two different fusing-braiding decompositions of each side of Eq.3.39, there are two 3D decompositions of the F3E polyhedron. This is represented in split view on the next figure, where the internal faces are hatched for clarity.



$$(3.41)$$

In general, the rule is to take a polyhedron with triangular faces and to decompose it in tetrahedra in different ways. Substituting the associated 6-j symbols yields the polynomial identities⁷. In quantum-group diagrams[20], a second world was introduced – the normal one – which is represented by solid lines. It corresponds to the ξ fields which we leave out of the present discussion.

⁷we restrict ourselves to polyhedra which are orientable surfaces.

4 Continuation to continuous J

The above discussion should be extended to non integer $2J$ in order to achieve a complete description of 2D gravity. This appears in several instances. First, the Liouville exponentials $e^{-J\alpha_- \Phi}$ with half integer J can be obtained from the discussion just recalled, but this does not allow to reach the field Φ itself. It may only be defined as $\Phi = -\frac{d}{dJ}|_{J=0} \exp(-J\alpha_- \Phi)/\alpha_-$, if continuous J may be handled. Second, in the strong coupling regime, values of J appear [19, 13] that are fourths or sixths of integers, in some cases. Treating these rational values is tantamount to going to the continuous case. The case of integer $2J$ corresponds, in the BPZ framework, to the appearance of degenerate fields satisfying null vector differential equations. For continuous J these equations are lacking. However, the structural analogy of $U_q(sl(2))$ with its classical counterpart together with the group-theoretical decomposition of the classical Liouville exponentials suggests that in the general situation, the chiral primaries should fall into one-sided highest/lowest weight representations of $U_q(sl(2))$. Furthermore one expects that they should obey a closed algebra under fusion and braiding, determined by the q-Clebsch-Gordan coefficients resp. R-matrix of $U_q(sl(2))$ relevant for these representations. By setting up a Coulomb-gas type representation for the chiral vertex operators with arbitrary real J , their exchange algebra becomes accessible to free field techniques, and we can prove that the braiding matrix is given by a natural analytic continuation of the positive half-integer spin case, defined in terms of Askey-Wilson polynomials. Then, the fusion matrix is determined by a generalization of the pentagonal equations discussed earlier. This part is a summary of refs. [11, 12].

4.1 The braiding

The purpose of this section is to derive that the operators $V_m^{(J)}$ ($\equiv V_{m,0}^{(J,0)}$) with $J + m = 0, 1, 2, \dots$ and continuous J , obey a closed exchange algebra. The classical expression Eq. 2.22 for S has a rather simple quantum generalization, which we will denote by the S to signify that it is a primary field of dimension zero (“screening charge”), namely [23]

$$S(\sigma) = e^{2ih(\varpi+1)} \int_0^\sigma d\rho V_1^{(-1)}(\rho) + \int_\sigma^{2\pi} d\rho V_1^{(-1)}(\rho) \quad (4.1)$$

Apart from an overall change of normalization — removal of the denominator — and the introduction of normal orderings, the only change consists in the replacement $\varpi \rightarrow \varpi + 1$ in the prefactor of the first integral. The quantum formula is such that S is periodic up to a multiplicative factor

$$S(\sigma + 2\pi) = e^{2ih(\varpi+1)} S(\sigma + 2\pi). \quad (4.2)$$

This is the quantum version of Eq.2.25. The basic primary field of the Coulomb gas picture is defined as

$$U_m^{(J)}(\sigma) = V_{-J}^{(J)}(\sigma)[S(\sigma)]^{J+m} \quad (4.3)$$

which is the quantum version of the first equality in Eq.2.24. The conformal dimension of $U_m^{(J)}$ agrees with Eq.3.9. Furthermore, one easily verifies that

$$U_m^{(J)} \varpi = (\varpi + 2m) U_m^{(J)} \quad (4.4)$$

which is the same as the first relation of Eqs.2.42. Since conformal weight and zero mode shift define a primary field uniquely up to a ϖ -dependent normalization, one certainly has

$$U_m^{(J)} = I_m^{(J)}(\varpi) V_m^{(J)} \quad (4.5)$$

The explicit expression for $I_m^{(J)}(\varpi)$ is

$$I_m^{(J)}(\varpi) = \left(2\pi \Gamma\left(1 + \frac{h}{\pi}\right) \right)^n e^{(ih(J+m)(\varpi-J+m))} \prod_{\ell=1}^{J+m} \frac{\Gamma[1 + (2J - \ell + 1)h/\pi]}{\Gamma[1 + \ell h/\pi] \Gamma[1 - (\varpi + 2m - \ell)h/\pi] \Gamma[1 + (\varpi + \ell)h/\pi]}. \quad (4.6)$$

This formula illustrates an important point to be made about the integral representation Eq.4.1. For small enough h , the arguments of the gamma functions are all positive, and this corresponds to the domain where the integral representation is convergent. When h increases, divergences appear. However, the equation just written continues to make sense provided none of the arguments of gamma functions precisely vanish or become a negative integer. This possibility is ruled out generically for continuous J and ϖ . Thus there exists a continuation of the integral representation that makes perfect

sense, and Eq.4.3 is meaningful for any h . We shall use the fields \tilde{V} instead of the V 's. The expression for the coupling constants g recalled above has an immediate extension to continuous J with $J + m$ integer, that is

$$g_{J, (\varpi - \varpi_0)/2+m}^{(\varpi - \varpi_0)/2} = \left(\frac{h}{\pi}\right)^{J+m} \times \prod_{k=1}^{J+m} \sqrt{\frac{F[1 + (2J - k + 1)h/\pi] F[(\varpi + 2m - k)h/\pi] F[-(\varpi + k)h/\pi]}{F[1 + kh/\pi]}}. \quad (4.7)$$

The treatment of the square roots require some care. We follow the prescription of ref.[24] also used in ref.[13]. Thus Eq.3.22 immediately extends to the case of non integer J : $\tilde{V}_m^{(J)} = g_{J, (\varpi - \varpi_0)/2+m}^{(\varpi - \varpi_0)/2} V_m^{(J)}$. The relation with $U_m^{(J)}$ is given by

$$U_m^{(J)} = \frac{I_m^{(J)}(\varpi)}{g_{J, (\varpi - \varpi_0)/2+m}^{(\varpi - \varpi_0)/2}} \tilde{V}_m^{(J)} \equiv \frac{1}{\kappa_{J, (\varpi - \varpi_0)/2+m}^{(\varpi - \varpi_0)/2}} \tilde{V}_m^{(J)}. \quad (4.8)$$

After some calculation, one finds

$$\kappa_{J_1 J_2}^{J_{12}} = \left(\frac{\pi e^{-i(h+\pi)}}{2\Gamma(1 + h/\pi) \sin h} \right)^{J_1 + J_2 - J_{12}} e^{ih(J_1 + J_2 - J_{12})(J_1 - J_2 - J_{12})} \times \prod_{k=1}^{J_1 + J_2 - J_{12}} \sqrt{\frac{[1 + 2J_1 - k]}{[k] [1 + 2J_2 - k] [1 + 2J_{12} + k]}}, \quad (4.9)$$

which makes sense for continuous J 's provided $J_1 + J_2 - J_{12}$ is a non-negative integer. Note that, for the fields $V_{-J}^{(J)}$, there is no distinction between $\tilde{V}_{-J}^{(J)}$ and $V_{-J}^{(J)}$, since $g_{J, x-J}^J = 1$. Hence, in Eqs.4.1, and 4.3, we may use \tilde{V} fields to represent normal ordered exponentials, so that only \tilde{V} fields appear in the discussion.

We now come to the braiding algebra of the fields $U_m^{(J)}$. We shall recall some basic points of the derivation following refs.[11, 12]. The braiding relation takes the form

$$U_m^{(J)}(\sigma) U_m^{(J')}(\sigma') = \sum_{m_1, m_2} R_U(J, J'; \varpi)_{m_2 m_1}^{m_2 m_1} U_{m_2}^{(J')}(\sigma') U_{m_1}^{(J)}(\sigma). \quad (4.10)$$

We only deal with the case $\pi > \sigma' > \sigma > 0$ explicitly. The other cases are deduced from the present one in the standard way. The sums extending over

non-negative integer $J + m_1$ resp. $J' + m_2$. with the condition

$$m_1 + m_2 = m + m' =: m_{12}. \quad (4.11)$$

Since one considers the braiding at equal τ one lets $\tau = 0$ once and for all, and works on the unit circle $u = e^{i\sigma}$. As there are no null-vector decoupling equations for continuous J , the derivation of Eq.4.10 is only based on the free field techniques summarized in section 2. The basic point of our derivation is that the exchange of two $U_m^{(J)}$ operators can be mapped into an equivalent problem in one-dimensional quantum mechanics, and becomes just finite-dimensional linear algebra. In view of Eq.4.1, 4.3, the essential observation is that one only needs the braiding relations of $\tilde{V}_{-J}^{(J)}$ operators which are normal ordered exponentials. (“tachyon operators”). One may verify that

$$\tilde{V}_{-J}^{(J)} = N^{[1]} \left(e^{2J\sqrt{h/2\pi}q^{[1]}} \right),$$

where we used the notation of section 2. This formula clearly makes sense for arbitrary J . An elementary computation gives

$$\tilde{V}_{-J}^{(J)}(\sigma)\tilde{V}_{-J'}^{(J')}(\sigma') = e^{-i2JJ'h\epsilon(\sigma-\sigma')}\tilde{V}_{-J'}^{(J')}(\sigma')\tilde{V}_{-J}^{(J)}(\sigma) \quad (4.12)$$

where $\epsilon(\sigma - \sigma')$ is the sign of $\sigma - \sigma'$. This means that when commuting the tachyon operators in $U_{m'}^{(J')}(\sigma')$ through those of $U_m^{(J)}(\sigma)$, one only encounters phase factors of the form $e^{\pm i2\alpha\beta h}$ resp. $e^{\pm 6i\alpha\beta h}$, with α equal to J or -1 , β equal to J' or -1 , since we take $\sigma, \sigma' \in [0, \pi]$. Hence we are led to decompose the integrals defining the screening charges S into pieces which commute with each other and with $\tilde{V}_{-J}^{(J)}(\sigma)$, $\tilde{V}_{-J'}^{(J')}(\sigma')$ up to one of the above phase factors. Let us write

$$\begin{aligned} S(\sigma) &= S_{\sigma\sigma'} + S_{\Delta}, \quad S(\sigma') = S_{\sigma\sigma'} + k(\varpi)S_{\Delta} \equiv S_{\sigma\sigma'} + \tilde{S}_{\Delta}, \\ S_{\sigma\sigma'} &:= k(\varpi) \int_0^{\sigma} \tilde{V}_1^{(-1)}(\rho)d\rho + \int_{\sigma'}^{2\pi} \tilde{V}_1^{(-1)}(\rho)d\rho, \\ S_{\Delta} &:= \int_{\sigma}^{\sigma'} \tilde{V}_1^{(-1)}(\rho)d\rho, \quad k(\varpi) := e^{2ih(\varpi+1)} \end{aligned} \quad (4.13)$$

Using Eq.4.12, we then get the following simple algebra for $S_{\sigma\sigma'}, S_{\Delta}, \tilde{S}_{\Delta}$:

$$S_{\sigma\sigma'}S_{\Delta} = q^{-2}S_{\Delta}S_{\sigma\sigma'}, \quad S_{\sigma\sigma'}\tilde{S}_{\Delta} = q^2\tilde{S}_{\Delta}S_{\sigma\sigma'}, \quad S_{\Delta}\tilde{S}_{\Delta} = q^4\tilde{S}_{\Delta}S_{\Delta}, \quad (4.14)$$

and their commutation properties with $\tilde{V}_{-J}^{(J)}(\sigma), \tilde{V}_{-J'}^{(J')}(\sigma')$ are given by

$$\begin{aligned} \tilde{V}_{-J}^{(J)}(\sigma)S_{\sigma\sigma'} &= q^{-2J}S_{\sigma\sigma'}\tilde{V}_{-J}^{(J)}(\sigma), & \tilde{V}_{-J'}^{(J')}(\sigma')S_{\sigma\sigma'} &= q^{-2J'}S_{\sigma\sigma'}\tilde{V}_{-J'}^{(J')}(\sigma'), \\ \tilde{V}_{-J}^{(J)}(\sigma)S_{\Delta} &= q^{-2J}S_{\Delta}\tilde{V}_{-J}^{(J)}(\sigma), & \tilde{V}_{-J}^{(J)}(\sigma)\tilde{S}_{\Delta} &= q^{-6J}\tilde{S}_{\Delta}\tilde{V}_{-J}^{(J)}(\sigma), \\ \tilde{V}_{-J'}^{(J')}(\sigma')S_{\Delta} &= q^{2J'}S_{\Delta}\tilde{V}_{-J'}^{(J')}(\sigma'), & \tilde{V}_{-J'}^{(J')}(\sigma')\tilde{S}_{\Delta} &= q^{-2J'}\tilde{S}_{\Delta}\tilde{V}_{-J'}^{(J')}(\sigma'). \end{aligned} \quad (4.15)$$

Finally, all three screening pieces obviously shift the zero mode in the same way:

$$\left. \begin{matrix} S_{\sigma\sigma'} \\ S_{\Delta} \\ \tilde{S}_{\Delta} \end{matrix} \right\} \varpi = (\varpi + 2) \left\{ \begin{matrix} S_{\sigma\sigma'} \\ S_{\Delta} \\ \tilde{S}_{\Delta} \end{matrix} \right. . \quad (4.16)$$

Using Eqs.4.15 we can commute $\tilde{V}_{-J}^{(J)}(\sigma)$ and $\tilde{V}_{-J'}^{(J')}(\sigma')$ to the left on both sides of Eq.4.10, so that they can be cancelled. Then we are left with

$$\begin{aligned} (q^{-2J'}S_{\Delta} + q^{2J'}S_{\sigma\sigma'})^{J+m}(\tilde{S}_{\Delta} + S_{\sigma\sigma'})^{J'+m'}q^{2JJ'} = \\ \sum_{m_1, m_2} R(J, J'; \varpi + 2(J + J'))_{m \ m'}^{m_2 m_1} (q^{2J}S_{\sigma\sigma'} + q^{6J}\tilde{S}_{\Delta})^{J'+m_2}(S_{\sigma\sigma'} + S_{\Delta})^{J+m_1} \end{aligned} \quad (4.17)$$

It is apparent from this equation that the braiding problem of the $U_m^{(J)}$ operators is governed by the Heisenberg-like algebra Eq.4.14, characteristic of one-dimensional quantum mechanics. However, to see this structure emerge, we had to decompose the screening charges $S(\sigma), S(\sigma')$ in a way which depends on both positions σ, σ' ; hence the embedding of this Heisenberg algebra into the 1+1 dimensional field theory is somewhat nontrivial. We shall proceed using the following simple representation of the algebra Eq.4.14 in terms of one-dimensional quantum mechanics (y and y' are arbitrary complex numbers):

$$S_{\sigma\sigma'} = y'e^{2Q}, \quad S_{\Delta} = ye^{2Q-P}, \quad \tilde{S}_{\Delta} = ye^{2Q+P}, \quad [Q, P] = ih. \quad (4.18)$$

The third relation in Eq.4.18 follows from the second one in view of $\tilde{S}_{\Delta} = k(\varpi)S_{\Delta}$ (cf. Eq.4.13). This means we are identifying here $P \equiv ih\varpi$ with the zero mode of the original problem. Using $e^{2Q+cP} = e^{cP}e^{2Q}q^c$ we can commute all factors e^{2Q} to the right on both sides of Eq.4.17 and then cancel them. This leaves us with

$$q^{2JJ'} \prod_{s=1}^{J+m} (y'q^{2J'} + yq^{-(\varpi-2J+2s-1)}) \prod_{t=1}^{J'+m'} (y' + yq^{\varpi-2J'+2m+2t-1}) =$$

$$\sum_{m_1} R_U(J, J'; \varpi)_{m m'}^{m_2 m_1} \prod_{t=1}^{J'+m_2} (y' q^{2J} + y q^{\varpi+4J-2J'+2t-1}) \prod_{s=1}^{J+m_1} (y' + y q^{-(\varpi-2J+2m_2+2s-1)}) \quad (4.19)$$

where we have shifted back $\varpi + 2(J + J') \rightarrow \varpi$ compared to Eq.4.17. Since the overall scaling $y \rightarrow \lambda y, y' \rightarrow \lambda y'$ only gives back Eq.4.11, we can set $y' = 1$.

The solution of these equations, which was derived in ref.[11], will be cast under the convenient form

$$R(J, J', \varpi)_{m m'}^{m_2 m_1} = e^{-i\pi(\Delta(c)+\Delta(b)-\Delta(e)-\Delta(f))} \frac{\kappa_{ab}^e \kappa_{de}^c}{\kappa_{db}^f \kappa_{af}^c} \left\{ \begin{matrix} a & b \\ d & c \end{matrix} \middle| \begin{matrix} e \\ f \end{matrix} \right\} \quad (4.20)$$

where

$$\begin{aligned} a &= J, & b &= x + m + m', & c &= x \equiv (\varpi - \varpi_0)/2 \\ d &= J', & e &= x + m_2, & f &= x + m. \end{aligned} \quad (4.21)$$

The r.h.s. may be expressed in terms of q-hypergeometric functions by the formula

$$\frac{\kappa_{ab}^e \kappa_{de}^c}{\kappa_{db}^f \kappa_{af}^c} \left\{ \begin{matrix} a & b \\ d & c \end{matrix} \middle| \begin{matrix} e \\ f \end{matrix} \right\}_q = \frac{[\rho]_{n'} [1-\beta]_{n-n_1} [\phi]_{n_1} [1-\epsilon-n_1]_{n_1}}{[n_1]! [\beta-\epsilon-n'-n]_{n+n'}} {}_4F_3 \left(\begin{matrix} \alpha, \beta, -n_1, -n' \\ \epsilon, \phi, \rho \end{matrix}; q, 1 \right) \quad (4.22)$$

where

$$\begin{aligned} \alpha &= -a - c + f, & \beta &= -c - d + e, \\ n_1 &= a + b - e, & n_2 &= d + e - c, & n &= f + a - c & n' &= b + d - f; \\ \epsilon &= -(a + b + c + d + 1), & \phi &= 1 + n - n_1, & \rho &= e + f - a - d + 1. \end{aligned} \quad (4.23)$$

Explicitly one has

$$n_1 = J + m_1, \quad n_2 = J + m_2, \quad n = J + m, \quad n' = J + m'. \quad (4.24)$$

Since they are equal to the screening numbers, they are positive integers. It follows that Eq.4.22 makes sense, since one defines, as in the previous work along the same line,

$${}_4F_3 \left(\begin{matrix} a, b, c, d \\ e, f, g \end{matrix}; q, \rho \right) = \sum_{n=0}^{\infty} \frac{[a]_n [b]_n [c]_n [d]_n}{[e]_n [f]_n [g]_n [n]!} \rho^n,$$

$$[a]_n := [a][a+1] \cdots [a+n-1], \quad [a]_0 := 1. \quad (4.25)$$

The method used to derive Eq.4.19 was to transform this hypergeometric function into another one such that the desired relations follow from the orthogonality relation of the associated Ashkey-Wilson polynomials. In this connection, let us simply note that a simple reshuffling of the parameters of the latter form, allows to verify that the usual orthogonality relations of the 6-j symbols extends to our case. One has, in general,

$$\sum_{J_{23}} \left\{ \begin{matrix} J_1 & J_2 \\ J_3 & J_{123} \end{matrix} \middle| \begin{matrix} J_{12} \\ J_{23} \end{matrix} \right\}_q \left\{ \begin{matrix} J_1 & J_2 \\ J_3 & J_{123} \end{matrix} \middle| \begin{matrix} K_{12} \\ J_{23} \end{matrix} \right\}_q = \delta_{J_{12}-K_{12}} \quad (4.26)$$

where the J 's are arbitrary chosen so that the screening numbers

$$\begin{aligned} n_1 &= J_1 + J_2 - J_{12}, & n_2 &= J_3 + J_{12} - J_{123}, & n &= J_1 + J_{23} - J_{123}, \\ n' &= J_2 + J_3 - J_{23}, & \tilde{n}_1 &= J_1 + J_2 - K_{12}, & \tilde{n}_2 &= J_3 + K_{12} - J_{123}, \end{aligned} \quad (4.27)$$

are positive integers. These conditions fix the range of summation over J_{23} .

4.2 Generalization of 6-j symbols

In this section, following ref.[13], we systematize the generalization of the 6-j symbols to non-half-integer spins, and prove the corresponding generalized polynomial equations⁸ in particular the pentagonal relation. This will allow to complete the picture of the OPA for continuous J , by including the fusion. For this, we note that relevant notion is not whether some spins are half-integer or not, but how many triangular inequalities are relaxed. Explicitly, we use dotted vertices, a dot on one leg meaning that the sum of the spins of the two other legs minus the spin of this leg is constrained to be a positive integer. Since there are three legs at a vertex we may have one two or three dots. We call them type TI1, TI2, TI3, where the letters T and I stands for triangular inequalities. A example of the TI1 case are

$$\begin{array}{c} \text{j}_1 \\ | \\ \text{j}_{12} \bullet \text{j}_2 \end{array} \rightarrow \text{j}_1 + \text{j}_2 - \text{j}_{12} \text{ positive integer}$$

⁸ This is an abusive use of the name “polynomial equations” which usually refers to consistency equations for fusion and braiding coefficients. But the 6-j coefficients, which are solutions of them, satisfy parallel equations, namely orthogonality, Racah identity, Biedenharn-Elliott identity..., that we generically call polynomial equations as well.

Adding dots to a vertex adds other restrictions. For the type TI3, we have

$$\begin{array}{c} J_1 \\ | \\ \underline{J_{12}} \bullet \bullet J_2 \end{array} \rightarrow \left\{ \begin{array}{ll} J_1 + J_2 - J_{12} & \text{positive integer} \\ J_{12} + J_2 - J_1 & \text{positive integer} \\ J_1 + J_{12} - J_2 & \text{positive integer} \end{array} \right\} \Rightarrow \begin{array}{l} 2J_1, 2J_2, 2J_{12} \\ \text{positive integers.} \end{array} \quad (4.28)$$

These are but the usual (full) triangular inequalities, or the branching rules for $U_q(sl(2))$, with half-integer spins. So, the standard operators are of the TI3 type.

The first step of generalization is to relax one restriction, which give the TI2 type:

$$\begin{array}{c} J_1 \\ | \\ \underline{j_{12}} \bullet \bullet j_2 \end{array} \rightarrow \left\{ \begin{array}{ll} J_1 + j_2 - j_{12} & \text{positive integer} \\ J_1 + j_{12} - j_2 & \text{positive integer} \end{array} \right\} \Rightarrow 2J_1 \text{ positive integer} \quad (4.29)$$

j_{12} and j_2 being arbitrary⁹. In this TI2 case, J_1 is a positive half-integer, and $j_2 - j_{12}$ is a half-integer (positive or negative). This is fixed by the number of restrictions. The second step is of course to keep only one restriction (type TI1):

$$\begin{array}{c} j_1 \\ | \\ \underline{j_{12}} \bullet \bullet j_2 \end{array} \rightarrow j_1 + j_2 - j_{12} \text{ positive integer} \quad (4.30)$$

and none of the spins are half-integers. Then the fusion or braiding of such generalized vertices lead to 6-j symbols generalized in a very specific way, and the “miracle” is that it is mathematically consistent. In the first step of generalization, fusion and braiding lead to two different generalized 6-j, whereas they lead to only one kind of 6-j in the second step. We define now generalized 6-j coefficients for the fusion and braiding of operators with only

⁹For the time being, we denote half-integer positive spins by capital letters and continuous ones by small letters, but this is only a consequence of the type of vertex and in no case an a priori assumption.

TI1 conditions. The diagrams are

$$\begin{array}{ccc}
 \begin{array}{c} |j_1 \\ \bullet \\ j \quad j+m_1 \quad j+m_{12} \end{array} & \longrightarrow & \begin{array}{c} j_{12} \quad |j_1 \quad j_2 \\ \bullet \\ j \quad j+m_{12} \end{array}
 \end{array} \quad (4.31)$$

$$\begin{array}{ccc}
 \begin{array}{c} |j_1 \quad |j_2 \\ \bullet \quad \bullet \\ j \quad j+m_1 \quad j+m_{12} \end{array} & \longrightarrow & \begin{array}{c} |j_2 \quad |j_1 \\ \bullet \quad \bullet \\ j \quad j+m'_2 \quad j+m_{12} \end{array}
 \end{array} \quad (4.32)$$

There are several equivalent forms, but the most convenient one, from the viewpoint of polynomial equations is

$$\begin{aligned}
 \left\{ \begin{array}{c} j_1 \quad j_2 \\ j_3 \quad j_{123} \end{array} \middle| \begin{array}{c} j_{12} \\ j_{23} \end{array} \right\}_q &= \frac{\Xi_{j_2 j_3}^{j_{23}} \Xi_{j_1 j_{23}}^{j_{123}}}{\Xi_{j_1 j_2}^{j_{12}} \Xi_{j_{12} j_3}^{j_{123}}} \frac{(-1)^{p_{12,3}+p_{1,2}} [p_{2,3}]! [p_{1,23}]! [2j_{23}+1]}{[j_{23}+j_{123}-j_1+1]_{p_{1,23}+p_{2,3}+1}} \\
 &\quad \sum_{p_{1,23}, p_{12,3} \leq y \leq p_{12,3}+p_{1,23}, p_{12,3}+p_{1,2}} (-1)^y [j_1+j_{23}+j_{123}+2]_{y-p_{1,23}} \\
 &\quad \frac{[y+2j_{123}-j_1-j_2-j_{12}+1]_{p_{12,3}+p_{1,23}-y}}{[y-p_{1,23}]! [y-p_{12,3}]!} \\
 &\quad \frac{[y+2j_{123}-j_2-j_3-j_{23}+1]_{p_{12,3}+p_{1,2}-y} [j_2+j_{12}+j_{23}-j_{123}-y+1]_{y-p_{12,3}}}{[p_{12,3}+p_{1,23}-y]! [p_{12,3}+p_{1,2}-y]!}
 \end{aligned} \quad (4.33)$$

where the p 's are such that

$$\begin{aligned}
 p_{1,2} &\equiv j_1 + j_2 - j_{12}, & p_{2,3} &\equiv j_2 + j_3 - j_{23}, \\
 p_{12,3} &\equiv j_{12} + j_3 - j_{123}, & p_{1,23} &\equiv j_1 + j_{23} - j_{123},
 \end{aligned}$$

so that

$$p_{k,l} \in \mathbb{Z}_+, \quad p_{1,2} + p_{12,3} = p_{1,23} + p_{2,3}. \quad (4.34)$$

For simplicity we lump the square roots of 6-j symbols into coefficients noted Ξ . Those Ξ factors are chosen so that they can be seen as normalization factors of the vertices and will cancel (or factorize) out of the pentagonal

equations when applying successive fusions (or braidings). So, the polynomial equations are fundamentally rational equations (without square roots). We define ($p_{1,2} \equiv j_1 + j_2 - j_{12}$)

$$\begin{aligned} \Xi_{j_1 j_2}^{j_{12}} &\equiv \prod_{k=1}^{p_{1,2}} \sqrt{\frac{[2j_1 - k + 1][2j_2 - k + 1][2j_{12} + k + 1]}{[k]}} \\ &= \sqrt{\frac{[j_1 - j_2 + j_{12} + 1]_{p_{1,2}} [-j_1 + j_2 + j_{12} + 1]_{p_{1,2}} [j_1 + j_2 + j_{12} + 1]_{p_{1,2}}}{[p_{1,2}]!}} \end{aligned} \quad (4.35)$$

We note that we only have the residual symmetry

$$\left\{ \begin{matrix} j_1 & j_2 & | & j_{12} \\ j_3 & j_{123} & | & j_{23} \end{matrix} \right\}_q = \left\{ \begin{matrix} j_3 & j_2 & | & j_{23} \\ j_1 & j_{123} & | & j_{12} \end{matrix} \right\}_q \quad (4.36)$$

The other symmetries are lost due to the particular choice of the quantities $p_{k,l}$ to be positive integers.

One may show that all the polynomial equations are obeyed by our generalization. thus we conclude that the fusion and braiding are still given by Eqs.3.12, 3.14, where all symbols are replaced by the generalized ones.

5 The case of two screening charges

Since this case is somewhat tedious, partly due to notational complications, we only give the results. The quantum group structure is now $U_q(sl(2)) \odot U_{\hat{q}}(sl(2))$, with $q = \exp(ih)$, and $\hat{q} = \exp(\hat{q})$, and h and \hat{h} are given by Eq.2.34.

It describes the fusion and braiding of the general operators $V_{m\hat{m}}^{(J\hat{J})}$ generated by the fusion and braiding of $V_{m0}^{(J0)} = V_m^{(J)}$, and $V_{0\hat{m}}^{(0\hat{J})} = V_{\hat{m}}^{(\hat{J})}$. For half integer spins, the \odot product is a sort of graded tensor product, since $V_m^{(J)}$ and $V_{\hat{m}}^{(\hat{J})}$ commute up to a phase. This is not true for continuous J where a completely novel structure emerges. The fusion and braiding of the general chiral operators $V_{\underline{m}}^{(\underline{J})}$, also denoted $V_{m\hat{m}}^{(J\hat{J})}$, where underlined symbols denote double indices $\underline{J} \equiv (J, \hat{J})$, $\underline{m} \equiv (m, \hat{m})$, takes the form

$$\mathcal{P}_{\underline{J}} V_{\underline{J}_{23}-\underline{J}}^{(\underline{J}_1)}(z_1) V_{\underline{J}_3-\underline{J}_{23}}^{(\underline{J}_2)}(z_2) = \sum_{\underline{J}_{12}} \frac{g_{\underline{J}_1 \underline{J}_2}^{\underline{J}_{12}} g_{\underline{J}_{12} \underline{J}_3}^{\underline{J}}}{g_{\underline{J}_2 \underline{J}_3}^{\underline{J}_{23}} g_{\underline{J}_1 \underline{J}_{23}}^{\underline{J}}} \left\{ \begin{matrix} \underline{J}_1 & \underline{J}_2 & | & \underline{J}_{12} \\ \underline{J}_3 & \underline{J} & | & \underline{J}_{23} \end{matrix} \right\}_q \times$$

$$\mathcal{P}_{\underline{J}} \sum_{\{\nu\}} V_{\underline{J}_3 - \underline{J}}^{(\underline{J}_{12}, \{\nu\})}(z_2) < \varpi_{\underline{J}_{12}}, \{\nu\} | V_{\underline{J}_2 - \underline{J}_{12}}^{(\underline{J}_1)} | \varpi_{\underline{J}_2}(z_1 - z_2) >. \quad (5.1)$$

$$\begin{aligned} \mathcal{P}_{\underline{J}} V_{\underline{J}_{23} - \underline{J}}^{(\underline{J}_1)}(z_1) V_{\underline{J}_3 - \underline{J}_{23}}^{(\underline{J}_2)}(z_2) &= \sum_{\underline{J}_{13}} e^{\pm i\pi(\Delta_{\underline{J}} + \Delta_{\underline{J}_3} - \Delta_{\underline{J}_{23}} - \Delta_{\underline{J}_{13}})} \times \\ &\frac{g_{\underline{J}_1 \underline{J}_3}^{\underline{J}_{13}}}{g_{\underline{J}_2 \underline{J}_3}^{\underline{J}_{23}}} \frac{g_{\underline{J}_{13} \underline{J}_3}^{\underline{J}}}{g_{\underline{J}_1 \underline{J}_{23}}^{\underline{J}}} \left\{ \begin{matrix} \underline{J}_1 & \underline{J}_3 \\ \underline{J}_2 & \underline{J} \end{matrix} \middle| \begin{matrix} \underline{J}_{12} \\ \underline{J}_{23} \end{matrix} \right\}_q \mathcal{P}_{\underline{J}} V_{\underline{J}_{13} - \underline{J}}^{(\underline{J}_2)}(z_2) V_{\underline{J}_3 - \underline{J}_{13}}^{(\underline{J}_1)}(z_1), \end{aligned} \quad (5.2)$$

In these formulae, the symbol $\varpi_{\underline{J}}$ stands for $\varpi_0 + 2J + 2\hat{J}\pi/h$ where $\varpi_0 = 1 + \pi/h$ corresponds to the $sl(2)$ -invariant vacuum; $\mathcal{P}_{\underline{J}}$ is the projector on $\mathcal{H}(\varpi_{\underline{J}})$. The above formulae contain the recoupling coefficients (q-6-J symbols) for the quantum group structure $U_q(sl(2)) \odot U_{\hat{q}}(sl(2))$. For half integer spins, they are defined by

$$\left\{ \begin{matrix} \underline{J}_1 & \underline{J}_2 \\ \underline{J}_3 & \underline{J} \end{matrix} \middle| \begin{matrix} \underline{J}_{12} \\ \underline{J}_{23} \end{matrix} \right\}_q = (-1)^{f_V(\underline{J}_1, \underline{J}_2, \underline{J}, \underline{J}_{23}, \underline{J}_3, \underline{J}_{12})} \left\{ \begin{matrix} J_1 & J_2 \\ J_3 & J \end{matrix} \middle| \begin{matrix} J_{12} \\ J_{23} \end{matrix} \right\}_q \left\{ \begin{matrix} \hat{J}_1 & \hat{J}_2 \\ \hat{J}_3 & \hat{J} \end{matrix} \middle| \begin{matrix} \hat{J}_{12} \\ \hat{J}_{23} \end{matrix} \right\}_{\hat{q}} \quad (5.3)$$

where $f_V(\underline{J}_1, \underline{J}_2, \underline{J}, \underline{J}_{23}, \underline{J}_3, \underline{J}_{12})$ is an integer given by

$$\begin{aligned} f_V(\underline{J}_1, \underline{J}_2, \underline{J}_{123}, \underline{J}_{23}, \underline{J}_3, \underline{J}_{12}) &= \\ 2\hat{J}_2(J_{12} + J_{23} - J_2 - J_{123}) + 2J_2(\hat{J}_{12} + \hat{J}_{23} - \hat{J}_2 - \hat{J}_{123}) \end{aligned} \quad (5.4)$$

This sign displays the grading associated with the \odot product for the half integer case. The symbol $\left\{ \begin{matrix} J_1 & J_2 \\ J_3 & J \end{matrix} \middle| \begin{matrix} J_{12} \\ J_{23} \end{matrix} \right\}_q$ is the 6j coefficient associated with $U_q(sl(2))$, while $\left\{ \begin{matrix} \hat{J}_1 & \hat{J}_2 \\ \hat{J}_3 & \hat{J} \end{matrix} \middle| \begin{matrix} \hat{J}_{12} \\ \hat{J}_{23} \end{matrix} \right\}_{\hat{q}}$ stands for the q-6j associated with $U_{\hat{q}}(sl(2))$. In addition to these group theoretic features there appear the coupling constants $g_{\underline{J}_1 \underline{J}_2}^{\underline{J}_{12}}$ whose general expression is given in ref[11].

For continuous spins, the J \hat{J} quantum numbers loose meaning and only the sum $J^e \equiv J + \hat{J}\pi/h$ is significant. Concerning the states, a similar phenomenon occurs. For a state associated with half-integer spins J and \hat{J} , the corresponding zero-mode is $\varpi = \varpi_{J, \hat{J}} \equiv \varpi_0 + 2J + 2\hat{J}\pi/h$. It is only a function of J^e and we have $\varpi_{J, \hat{J}} = \varpi_0 + 2J^e$, also denoted ϖ_{J^e} . One may verify that the fusion and braiding matrices may be written in terms of these effective spins. Of course, if h is irrational, using J , \hat{J} or J^e is immaterial. In the half-integer spin case, there is a remarkable fact, which is related: by using the properties of the gamma functions under integer shifts and of the sinus functions under shifts by $(\pi \times \text{integer})$ one may actually absorb the sign

factors into the 6-j's. It is this last form which carries over to the continuous case, where one has, for instance,

$$F_{J_{23}^e, J_{12}^e} \left[\begin{matrix} J_1^e & J_2^e \\ J_{123}^e & J_3^e \end{matrix} \right] = \frac{g_{J_1^e J_2^e}^{J_{12}^e} g_{J_{123}^e J_3^e}^{J_{123}^e}}{g_{J_2^e J_3^e}^{J_{23}^e} g_{J_1^e J_{23}^e}^{J_{123}^e}} \left\{ \left\{ \begin{matrix} J_1^e & J_2^e \\ J_3^e & J_{123}^e \end{matrix} \middle| \begin{matrix} J_{12}^e \\ J_{23}^e \end{matrix} \right\} \right\} \left\{ \left\{ \begin{matrix} \hat{J}_1^e & \hat{J}_2^e \\ \hat{J}_3^e & \hat{J}_{123}^e \end{matrix} \middle| \begin{matrix} \hat{J}_{12}^e \\ \hat{J}_{23}^e \end{matrix} \right\} \right\}. \quad (5.5)$$

the 6-j symbols with double braces are ordinary symbols with shifted arguments:

$$\left\{ \left\{ \begin{matrix} J_1^e & J_2^e \\ J_3^e & J_{123}^e \end{matrix} \middle| \begin{matrix} J_{12}^e \\ J_{23}^e \end{matrix} \right\} \right\} \equiv \left\{ \begin{matrix} J_1^e & J_2^e \\ J_3^e & J_{123}^e + (\hat{p}_{2,3} + \hat{p}_{1,23})\pi/h \end{matrix} \middle| \begin{matrix} J_{12}^e + \hat{p}_{1,2}\pi/h \\ J_{23}^e + \hat{p}_{2,3}\pi/h \end{matrix} \right\}_q, \quad (5.6)$$

with similar relations for the hatted ones. It is only for half integer J that one may disentangle the hatted and unhatted quantum numbers.

6 Applications

As is well know there are two completely different regimes. The explicit formulae for the quantum group parameters are

$$\begin{aligned} h &= \frac{\pi}{12} \left(C - 13 - \sqrt{(C - 25)(C - 1)} \right), \\ \hat{h} &= \frac{\pi}{12} \left(C - 13 + \sqrt{(C - 25)(C - 1)} \right), \end{aligned} \quad (6.1)$$

The weak coupling regime ($C \geq 25$, and $C \leq 1$) is such that h and \hat{h} are real. In the strong coupling one ($1 \leq C \leq 25$), on the contrary, h and \hat{h} are complex (conjugate).

6.1 The weak coupling regime

Here the point is to reconstruct the Liouville exponentials so that locality is ensured. The orthogonality relations Eqs.4.26 are precisely what we need for this purpose. Indeed we shall write, for arbitray J ,

$$e^{-J\alpha_- \Phi(\sigma, \tau)} = \sum_{m=-J}^{\infty} \tilde{V}_m^{(J)}(z) \overline{\tilde{V}}_m^{(J)}(\bar{z}) \quad (6.2)$$

We assume that $\tilde{V}_m^{(J)}(z)$, and $\overline{\tilde{V}}_m^{(J)}(\bar{z})$ commute. According to Eq.4.8, the braiding of the \tilde{V} fields is equal to the 6-j symbols. Thus the braiding matrix of the \tilde{V} fields is the same as the one of the $\overline{\tilde{V}}$ fields, and we have

$$\begin{aligned}
& e^{-J_1\alpha_-\Phi(\sigma_1,\tau)} e^{-J_2\alpha_-\Phi(\sigma_2,\tau)} = \\
& \sum_{m_1,m_2} \sum_{n_1,n_2;\bar{n}_1,\bar{n}_2} \left\{ \begin{matrix} J_1 & (\varpi-\varpi_0+2m_1+2m_2)/2 \\ J_2 & (\varpi-\varpi_0)/2 \end{matrix} \middle| \begin{matrix} (\varpi-\varpi_0+2n_2)/2 \\ (\varpi-\varpi_0+2m_1)/2 \end{matrix} \right\}_q \\
& \times \left\{ \begin{matrix} J_1 & (\overline{\varpi}-\varpi_0+2m_1+2m_2)/2 \\ J_2 & (\overline{\varpi}-\varpi_0)/2 \end{matrix} \middle| \begin{matrix} (\overline{\varpi}-\varpi_0+2\bar{n}_2)/2 \\ (\overline{\varpi}-\varpi_0+2m_1)/2 \end{matrix} \right\}_q \\
& \times \tilde{V}_{n_2}^{(J_2)}(z_2) \tilde{V}_{n_1}^{(J_1)}(z_1) \overline{\tilde{V}}_{n_2}^{(J_2)}(\bar{z}_2) \overline{\tilde{V}}_{n_1}^{(J_1)}(\bar{z}_1) \Big|_{\substack{n_1+n_2=m_1+m_2 \\ \bar{n}_1+\bar{n}_2=m_1+m_2}}. \quad (6.3)
\end{aligned}$$

We only discuss the case where $\varpi = \overline{\varpi}$ (no winding number) for simplicity. Then, the summation over m_1 in the last equation precisely coincides with the summation over J_{23} in Eq.4.26. This gives immediately

$$e^{-J_1\alpha_-\Phi(\sigma_1,\tau)} e^{-J_2\alpha_-\Phi(\sigma_2,\tau)} = e^{-J_2\alpha_-\Phi(\sigma_2,\tau)} e^{-J_1\alpha_-\Phi(\sigma_1,\tau)}, \quad (6.4)$$

and the Liouville exponential is local for arbitrary J . Since we have defined the Liouville exponential for countinuous J we may obtain the Liouville field itself by computing

$$\Phi(\sigma,\tau) \equiv -\frac{1}{\alpha_-} \frac{de^{-J\alpha_-\Phi(\sigma,\tau)}}{dJ} \Big|_{J=0} \quad (6.5)$$

One may verify[12] that the canonical commutation relations hold:

$$[\Phi(\sigma,\tau)\dot{\Phi}(\sigma',\tau)] = 4\pi i\gamma\delta(\sigma-\sigma'), \quad (6.6)$$

and that the quantum Liouville equation is satisfied.

So far the only direct application of the weak coupling formulae is the derivation of the matrix-model results on the sphere given in ref.[24]. One couples the Liouville theory with another copy of the same theory with central charge $c = 26 - C$, which represents matter. For $C > 25$, we have $c < 1$. The resulting three-point function on the sphere is a product of leg factors as expected.

6.2 The strong coupling regime

The problem is the so called “ $c = 1$ barrier”. The present approach is the only way[8, 13, 19] to go through it. The basic reason seems to be that one should treat the two screening charges symmetrically in the strong coupling regime, since they are complex conjugate. This is in sharp contrast with what is currently done in the weak coupling regime, say using matrix models. In the strong coupling regime $1 \leq C \leq 25$, h and \hat{h} are complex conjugate. Thus, treating them symmetrically, as was done in refs.[8, 13, 19], is the key. The basic result is the derivation of truncation theorems that hold for special values of C . The point of these theorems is as follows. The basic family of $V_{m\hat{m}}^{(J\hat{J})}$ operators has Virasoro weights given by

$$\Delta_{J\hat{J}} = \frac{C-1}{24} - \frac{1}{24} \left((J + \hat{J} + 1)\sqrt{C-1} - (J - \hat{J})\sqrt{C-25} \right)^2, \quad (6.7)$$

in agreement with Kac’s formula. In the strong coupling regime $\sqrt{C-1}$ is real, and $\sqrt{C-25}$ pure imaginary so that the formula just written give complex results in general. However—in a way that is reminiscent of the truncations that give the minimal unitary models—for $C = 7, 13$, and 19, there is a consistent truncation of the above general family down to an operator algebra involving operators with real Virasoro conformal weights only. These are of two types. The first has spins $\hat{J} = J$, and Virasoro weights that are negative; the second has $\hat{J} = -J - 1$, and Virasoro weights that are positive. the truncation theorems make use of the following notions.

a) The physical Hilbert spaces. They are of the form

$$\mathcal{H}_{s\text{phys}}^{\pm} \equiv \bigoplus_{r=0}^{1 \mp s} \bigoplus_{n=-\infty}^{\infty} \mathcal{H}_s(\varpi_{r,n}^{\pm}), \quad (6.8)$$

$$\varpi_{r,n}^{\pm} \equiv \left(\frac{r}{2 \mp s} + n \right) \left(1 \mp \frac{\pi}{h} \right). \quad (6.9)$$

The integer s is such that the special values correspond to

$$C = 1 + 6(s+2), \quad s = 0, \pm 1; \quad h + \hat{h} = s\pi. \quad (6.10)$$

For the plus sign, the weight $\Delta(\varpi_{r,n}^+) \equiv (1 + \pi/h)^2 h/4\pi - h\varpi_{r,n}^2/4\pi$ is positive and in $\mathcal{H}_{s\text{phys}}^+$, the representation of the Virasoro algebra is unitary. For

the minus sign $\Delta(\varpi_{r,n}^-)$ is real but negative. This latter case is only useful for topological theories. In $\mathcal{H}_{s\text{phys}}^+$ the partition function corresponds to compactification on a circle with radius $R = \sqrt{2(2-s)}$ (see refs.[13]).

b) The restricted set of conformal weights. The truncated family only involves operators of the type $(2J+1, 2J+1)$ noted $\chi_-^{(J)}$ and $(-2J-1, 2J+1)$ noted $\chi_+^{(J)}$. Their Virasoro conformal weights[8, 19, 13] which are respectively given by

$$\Delta^-(J, C) = -\frac{C-1}{6} J(J+1), \quad \Delta^+(J, C) = 1 + \frac{25-C}{6} J(J+1), \quad (6.11)$$

are real. $\Delta^-(J)$ is negative for all J (except for $J = -1/2$ where it becomes equal to $\Delta^+(-1/2) = (s+2)/4$). $\Delta^+(J)$ is always positive, and is larger than one if $J \neq -1/2$.

c) The truncated families: \mathcal{A}_{phys}^\pm is the set of operators noted $\chi_\pm^{(J)}$, introduced in [8, 19, 13], whose conformal weights are given by Eq.6.11. they are of the form

$$\mathcal{P}_{J_{12}^{e-}} \chi_-^{(J_1)} \equiv \sum_{J_2, p_1, 2 \in Z_+} (-1)^{(2+s)(2J_2 p_1, 2 + \frac{p_1, 2(p_1, 2+1)}{2})} g_{J_1^{e-} J_2^{e-}}^{J_{12}^{e-}} \mathcal{P}_{J_{12}^{e-}} V^{(J_1^{e-})} \mathcal{P}_{J_2^{e-}} \quad (6.12)$$

$$\mathcal{P}_{J_{12}^{e+}} \chi_+^{(J_1)} \equiv \sum_{J_2, p_1, 2 \in Z_+} (-1)^{(2-s)(2J_2 p_1, 2 + \frac{p_1, 2(p_1, 2+1)}{2})} g_{J_1^{e+} J_2^{e+}}^{J_{12}^{e+}} \mathcal{P}_{J_{12}^{e+}} V^{(J_1^{e+})} \mathcal{P}_{J_2^{e+}}. \quad (6.13)$$

This definition is such that the following holds

THE TRUNCATION THEOREMS:

For $C = 1+6(s+2)$, $s = 0, \pm 1$, and when it acts on $\mathcal{H}_{s\text{phys}}^+$ (resp. $\mathcal{H}_{s\text{phys}}^-$); the set \mathcal{A}_{phys}^+ (resp. \mathcal{A}_{phys}^-) of operators $\chi_+^{(J)}$ (resp. $\chi_-^{(J)}$) is closed by fusion and braiding, and only gives states that belong to $\mathcal{H}_{s\text{phys}}^+$ (resp. $\mathcal{H}_{s\text{phys}}^-$).

Let us now turn to the new topological models put forward in ref.[19]. One considers two copies of the above strongly coupled theories with central charges $C = 1 + 6(s+2)$, and $c = 1 + 6(-s+2)$. They play the role of gravity and matter respectively. This gives consistent theories since clearly $C + c = 26$. As shown in ref.[13], the three-point functions are calculable and given by a product of g factors of the type Eq.4.7. The result is again a product of leg factors, and the higher point functions seem calculable using

methods similar to the one used for the weak coupling regime. The novel feature is that now the vertex operators are expressed in terms of the chi fields given by Eqs.6.12 6.13 — and their antiholomorphic counterparts — instead of the Liouville exponentials. As a result they do not preserve any longer the equality between left- and right-quantum group spins. Thus the transition through the $c = 1$ barrier seems to be characterized by a sort of deconfinement of chirality.

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